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Abel-type Summability

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ABEL-TYPE SUMMABILITY

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ABSTRACT

The Abel method A is one of the fundamental methods of summability. It has been generalized in two directions to Abel-type methods A_λ and A'_λ by Borwein in his papers which appeared in 1957 and 1960. In this thesis the absolute and strong summability methods based upon the Abel-type methods A_λ and A'_λ are considered. Some inclusion theorems for the absolute and the strong summability for both scales are obtained. The relationship between the scales of methods A_λ and A'_λ for the absolute and the strong summability is investigated and basic equivalence theorems are proved. The translativity of the absolute Abel-type methods and the absolute logarithmic method is proved. The product of Abel-type methods with regular Hausdorff methods is considered and inclusion theorems for both the absolute and the strong summability are obtained. The absolute logarithmic method and its product with a regular Hausdorff method are also investigated. Finally, some applications of the results obtained are given, thereby obtaining the absolute summability analogues of known results.

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TABLE OF CONTENTS

ABSTRACT....................iii

ACKNOWLEDGEMENTS.................... iv

CHAPTER 1. Introduction.................... 1

CHAPTER 2. Absolute Abel-type Summability.................... 10

CHAPTER 3. Strong Abel-type Summability-I.................... 29

CHAPTER 4. Strong Abel-type Summability-II.................... 45

CHAPTER 5. The Product Method $A_\lambda H_\chi$.................... 64

CHAPTER 6. The Logarithmic Method L.................... 75

CHAPTER 7. Some Applications.................... 92

REFERENCES....................102

VITA....................109

CHAPTER 1

INTRODUCTION

It is supposed throughout the thesis that $\sum_{n=0}^{\infty} u_n$ is a given series with $\{s_n\}$ as the associated sequence of its partial sums, i.e.,

$$s_n = \sum_{r=0}^{n} u_r , \quad n = 0, 1, 2, \ldots .$$

The symbol M is used throughout the thesis to denote a positive number, independent of the variables under consideration, but not necessarily having the same value at each occurrence.

The theorems and lemmas in the thesis are numbered chapterwise, e.g., Theorem 5.1 is the first theorem in Chapter 5. The relations are numbered according to the section and chapter in which they occur, e.g., (3.2.5) is the fifth relation in section 2 of Chapter 3.

1.1 *SUMMABILITY METHODS*

If a summability method P assigns the value ℓ to the series $\sum_{n=o}^{\infty} u_n$, we say that $\sum_{n=o}^{\infty} u_n$ is P-*summable* to the *sum* ℓ and write

$$\sum_{n=o}^{\infty} u_n = \ell \ (P) \ .$$

We shall also say that the sequence $\{s_n\}$ is P-*convergent* to the *limit* ℓ and write

$$s_n \to \ell \ (P) \ .$$

A method of summability P is said to be *regular* if $s_n \to \ell$ (P) whenever the sequence $\{s_n\}$ converges to ℓ in the ordinary sense.

If P and Q are two methods of summability, P is said to *include* Q (written "$Q \Rightarrow P$") if every sequence convergent by the Q method is also convergent by the P method to the same limit. If $P \Rightarrow Q$ and $Q \Rightarrow P$, the two methods P and Q are said to be *equivalent* and we write $P \simeq Q$.

A summability method P is called *translative* if $s_n \to \ell$ (P) is equivalent to $s_{n+1} \to \ell$ (P).

1.2 *ABEL-TYPE METHODS*

The following notations are used throughout the thesis:

Let λ be any real number, and

$$\varepsilon_n^\lambda = \binom{n+\lambda}{n} = \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+n)}{n!}, \quad n = 0, 1, \ldots .$$

$$s_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n ,$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda u_n \left(\frac{y}{1+y}\right)^n ,$$

$$U_\lambda(y) = \lambda \int_o^y u_\lambda(t)dt ,$$

$$t_n = nu_n , \qquad n = 0, 1, \ldots$$

$$t_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda t_n \left(\frac{y}{1+y}\right)^n .$$

It is easily shown that

$$(1.2.1) \qquad y\frac{d}{dy} s_\lambda(y) = (\lambda+1)[s_{\lambda+1}(y) - s_\lambda(y)] = t_\lambda(y)$$

The Abel-type summability methods A_λ and A'_λ were introduced by Borwein [4] and [9] and are defined as follows:

The Abel-type summability method A_λ.

If

$$(1-x)^{\lambda+1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n x^n$$

is convergent for all x in the open interval (0, 1) and tends to a finite limit ℓ as $x \to 1$ in (0, 1), we say that the sequence $\{s_n\}$ is A_λ-*convergent* to ℓ and write

$$s_n \to \ell \ (A_\lambda) .$$

Evidently, $s_n \to \ell\ (A_\lambda)$ if and only if the series defining $s_\lambda(y)$ is convergent for all $y > 0$ and $s_\lambda(y) \to \ell$ as $y \to \infty$. The method A_o is the ordinary Abel method A.

The Abel-type summability method A'_λ.

If $u_\lambda(y)$ is defined for all $y > 0$ and $U_\lambda(y)$ tends to a finite limit ℓ as $y \to \infty$, we say that the sequence $\{s_n\}$ is A'_λ-*convergent* to ℓ and write

$$s_n \to \ell\ (A'_\lambda)\ .$$

1.3 *REGULARITY AND INCLUSION THEOREMS FOR ABEL-TYPE METHODS*.

In this section the regularity and inclusion properties of Abel-type methods are discussed. For the sake of completeness, proofs of some known results are also given.

THEOREM 1.1

The method A_λ is regular for $\lambda > -1$.

PROOF.

Suppose that $s_n \to \ell$. Since

$$(1.3.1) \qquad (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda \left(\frac{y}{1+y}\right)^n = 1,$$

there is no loss of generality in assuming $\ell = 0$. Let m be any positive integer. Then

$$s_\lambda(y) = (1+y)^{-\lambda-1}\sum_{n=0}^{m-1} \varepsilon_n^\lambda s_n\left(\frac{y}{1+y}\right)^n + (1+y)^{-\lambda-1}\sum_{n=m}^{\infty} \varepsilon_n^\lambda s_n\left(\frac{y}{1+y}\right)^n ,$$

and so,

$$\overline{\lim_{y\to\infty}}|s_\lambda(y)| \leq \overline{\lim_{y\to\infty}}\left\{(1+y)^{-\lambda-1}\sum_{n=0}^{m-1} \varepsilon_n^\lambda |s_n| \left(\frac{y}{1+y}\right)^n\right\}$$

$$+ \overline{\lim_{y\to\infty}}\left\{(1+y)^{-\lambda-1}\sum_{n=m}^{\infty} \varepsilon_n^\lambda |s_n| \left(\frac{y}{1+y}\right)^n\right\}$$

$$= \overline{\lim_{y\to\infty}}\left\{(1+y)^{-\lambda-1}\sum_{n=m}^{\infty} \varepsilon_n^\lambda |s_n| \left(\frac{y}{1+y}\right)^n\right\}$$

$$\leq \sup_{n\geq m}|s_n| \overline{\lim_{y\to\infty}}\left\{(1+y)^{-\lambda-1}\sum_{n=0}^{\infty} \varepsilon_n^\lambda \left(\frac{y}{1+y}\right)^n\right\} .$$

It follows by (1.3.1) that

$$\overline{\lim_{y\to\infty}} |s_\lambda(y)| \leq \sup_{n\geq m}|s_n| .$$

Hence

$$\lim_{y\to\infty} s_\lambda(y) = 0,$$

i.e.,

$$s_n \to 0 \ (A_\lambda) .$$

The theorem is proved.

It is easily seen by putting $x = \frac{y}{1+y}$, that for $\lambda > 0$, A'_λ is a moment constant method with $\mu_n = \frac{1}{\varepsilon_n^\lambda}$ and the generating mass function $\chi(t) = -(1-t)^\lambda$.

Hence, it follows as a consequence of Hardy [14] (Theorem 34) that:

THEOREM 1.2

The method A'_λ *is regular for* $\lambda > 0$.

The next lemma is easy and will be used extensively in the thesis without any further comment.

LEMMA 1.1

If $\lambda > -1$, $\mu > -1$, *then the series* $\sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n x^n$ *is convergent for* $|x| < 1$ *if and only if the series* $\sum_{n=o}^{\infty} \varepsilon_n^\mu u_n x^n$ *is convergent for* $|x| < 1$.

PROOF.

Since

$$\varepsilon_n^\lambda \sim \frac{n^\lambda}{\Gamma(\lambda+1)} \quad \text{as} \quad n \to \infty,$$

we have that

$$\lim_{n\to\infty} |\varepsilon_n^\lambda|^{\frac{1}{n}} = 1 .$$

It follows that the radius of convergence of $\sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n x^n$ is equal to the radius of convergence of $\sum_{n=o}^{\infty} s_n x^n$ and the radius of convergence of $\sum_{n=o}^{\infty} \varepsilon_n^\mu u_n x^n$ is equal to the radius of convergence of $\sum_{n=o}^{\infty} u_n x^n$.

Now, suppose that $\sum_{n=o}^{\infty} s_n x^n$ converges for $|x| < 1$. Then

$$\sum_{n=o}^{\infty} u_n x^n = \sum_{n=o}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_n x^{n+1} ,$$

and so $\sum_{n=o}^{\infty} u_n x^n$ is convergent for $|x| < 1$.

Conversely, suppose that $\sum_{n=o}^{\infty} u_n x^n$ is convergent for $|x| < 1$.

Since $\sum_{n=o}^{\infty} x^n$ is convergent to $\frac{1}{1-x}$ for $|x| < 1$, we have

$$\sum_{n=o}^{\infty} s_n x^n = \sum_{n=o}^{\infty} x^n \sum_{n=o}^{\infty} u_n x^n = \frac{1}{1-x} \sum_{n=o}^{\infty} u_n x^n ,$$

Hence $\sum_{n=o}^{\infty} s_n x^n$ is convergent for $|x| < 1$.

This completes the proof of the lemma.

The following lemma is due to Borwein [4].

LEMMA 1.2

If $\lambda > \mu > -1$, $y > 0$ *and* $\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{t}{1+t}\right)^n$ *is convergent for all* $t > 0$, *then*

$$(1.3.2) \qquad s_{\mu}(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} y^{-\lambda} \int_o^y (y-t)^{\lambda-\mu-1} t^{\mu} s_{\lambda}(t)dt.$$

PROOF.

By Lemma 1.1, the series $\sum_{n=o}^{\infty} \varepsilon_n^{\mu} s_n \left(\frac{t}{1+t}\right)^n$ is absolutely convergent for all $t > 0$ and so we can carry out term by term integration to get,

$$(1.3.3) \qquad y^{-\lambda} \int_o^y (y-t)^{\lambda-\mu-1} t^{\mu} s_{\lambda}(t)dt$$

$$= \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n y^{-\lambda} \int_o^y (y-t)^{\lambda-\mu-1} t^{\mu+n} (1+t)^{-\lambda-1-n} dt.$$

Now, putting

$$u = \frac{t}{1+t}, \quad x = \frac{y}{1+y},$$

we have

$$\int_0^y (y-t)^{\lambda-\mu-1} t^{\mu+n} (1+t)^{-\lambda-1-n} dt = (1-x)^{1+\mu-\lambda} \int_0^x (x-u)^{\lambda-\mu-1} u^{\mu+n} \, du$$

$$= x^{\lambda+n} (1-x)^{1+\mu-\lambda} \frac{\Gamma(\lambda-\mu)\Gamma(\mu+1+n)}{\Gamma(\lambda+1+n)} .$$

Hence,

$$\text{(1.3.4)} \qquad \varepsilon_n^\lambda y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} \, t^{\mu+n} \, (1+t)^{-\lambda-1-n} \, dt$$

$$= (1+y)^{-\mu-1} \left(\frac{y}{1+y}\right)^n \varepsilon_n^\mu \frac{\Gamma(\lambda-\mu)\Gamma(\mu+1)}{\Gamma(\lambda+1)} .$$

It follows from (1.3.3) and (1.3.4) that

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)} \; y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} \, t^\mu s_\lambda(t) dt$$

$$= (1+y)^{-\mu-1} \sum_{n=0}^{\infty} \varepsilon_n^\mu s_n \left(\frac{y}{1+y}\right)^n$$

$$= s_\mu(y).$$

This establishes (1.3.2) and thus the lemma is proved.

The transformation given by (1.3.2) is a regular transformation. It follows as a consequence of Hardy [14](Theorem 6)that:

THEOREM 1.3

If $\lambda > \mu > -1$ *and* $s_n \to \ell\ (A_\lambda)$, *then* $s_n \to \ell\ (A_\mu)$.

The following two theorems were proved by Borwein [9].

THEOREM A

For $\lambda > 0$, $s_n \to \ell\ (A_\lambda)$ *if and only if* $s_n \to \ell\ (A'_\lambda)$ *and* $nu_n \to 0\ (A_{\lambda-1})$.

THEOREM B

For $\lambda > 0$, $A'_\lambda \approx A_{\lambda-1}$.

These two theorems play a central role in the following three chapters, where the absolute and strong summability analogues are established.

CHAPTER 2

ABSOLUTE ABEL-TYPE SUMMABILITY

In this chapter absolute summability methods based upon the Abel-type methods A_λ and A'_λ are considered and some of their properties investigated. The absolute Abel-summability $|A|$ was first defined by J.M. Whittaker [33], who also proved that $|C, 0| \Rightarrow |A|$. It has been subsequently investigated by various authors: e.g., T.M. Flett [12] has given a generalization to the summability $|A|_k$. The absolute Abel-type summability $|A_\lambda|$ has been studied by B.P. Mishra [23].

The main results proved in this chapter are Theorems 2.4 and 2.5, which provide the absolute summability analogues of Theorems A and B (Chapter 1) respectively.

2.1 *DEFINITIONS*

Let $s_\lambda(y)$, $u_\lambda(y)$ and $U_\lambda(y)$ be as defined in §1.2.

Absolute Abel-type summability $|A_\lambda|$.

If $s_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit ℓ as $y \to \infty$, we say that the sequence $\{s_n\}$ is *absolutely* A_λ*-convergent* or $|A_\lambda|$*-convergent* to ℓ and write

$$s_n \to \ell \ |A_\lambda|.$$

Absolute Abel-type summability $|A'_\lambda|$.

If $U_\lambda(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit ℓ as $y \to \infty$, we say that the sequence $\{s_n\}$ is *absolutely* A'_λ*-convergent* or $|A'_\lambda|$*-convergent* to ℓ and write

$$s_n \to \ell \ |A'_\lambda|.$$

Before proving any results, we make the following remarks.

REMARKS

1. The function $\psi(x) = (1-x)^{\lambda+1} \sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n x^n$ is of bounded variation in $[0, 1)$ if and only if $s_\lambda(y)$ is of bounded variation in $[0, \infty)$.

For, if $y = g(x) = \frac{x}{1-x}$, then g is monotonic in $[0, 1)$ and g^{-1} is monotonic in $[0, \infty)$. Since a monotonic function of a monotonic function is monotonic, the result follows.

We shall take advantage of this remark without any further comment.

2. A function $f(x)$ of bounded variation in $[0, \infty)$ necessarily tends to a finite limit as $x \to \infty$.

(See Natanson [25] p. 239, Corollary to Theorem 5).

2.2 *PRELIMINARY RESULTS*

In this section we prove three lemmas which will be required in the following section. The first lemma contains identities involving the transforms $s_\lambda(y)$, $u_\lambda(y)$ and $U_\lambda(y)$. Some of these identities have, in essence, been proved by Borwein [9], pp. 73-74. For the sake of continuity in argument, complete proofs are given here.

LEMMA 2.1

If $\lambda > -1$, $y > 0$ *and* $\sum_{n=o}^{\infty} \varepsilon_n^\lambda s_n\left(\frac{t}{1+t}\right)^n$ *converges for all* $t > 0$, *then*

(2.2.1) $$u_\lambda(y) = (1+y)^{-1}s_\lambda(y) - \lambda(1+y)^{-\lambda-1}\int_o^y (1+t)^{\lambda-1}s_\lambda(t)dt.$$

(2.2.2) $$u_\lambda(y) = (1+y)^{-\lambda-1}s_\lambda(0) + (1+y)^{-\lambda-1}\int_o^y (1+t)^\lambda s_\lambda'(t)dt.$$

(2.2.3) $$U_\lambda(y) = \lambda(1+y)^{-\lambda}\int_o^y (1+t)^{\lambda-1}s_\lambda(t)dt.$$

(2.2.4) $$s_\lambda(y) = U_\lambda(y) + (1+y)u_\lambda(y).$$

(2.2.5) $$s_\lambda(y) = U_{\lambda+1}(y) + u_\lambda(y).$$

(2.2.6) $$yu_\lambda(y) = U_{\lambda+1}(y) - U_\lambda(y).$$

(2.2.7) $$U_{\lambda+1}(y) = (\lambda+1)y^{-\lambda-1}\int_o^y t^\lambda U_{\lambda+2}(t)dt.$$

PROOF.

Since

$$\varepsilon_{n+1}^\lambda = \frac{n+1+\lambda}{n+1}\varepsilon_n^\lambda ,$$

and $\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{y}{1+y}\right)^n$ is convergent for $y > 0$, we have

$$u_{\lambda}(y) = (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda}\ u_n \left(\frac{y}{1+y}\right)^n$$

$$= (1+y)^{-\lambda-1}\left[\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{y}{1+y}\right)^n - \sum_{n=o}^{\infty} \varepsilon_{n+1}^{\lambda} s_n \left(\frac{y}{1+y}\right)^{n+1}\right]$$

$$= (1+y)^{-\lambda-1}\left[\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{y}{1+y}\right)^n - \sum_{n=o}^{\infty} \left(1 + \frac{\lambda}{n+1}\right)\varepsilon_n^{\lambda} s_n \left(\frac{y}{1+y}\right)^{n+1}\right]$$

$$= s_{\lambda}(y)\left(1 - \frac{y}{1+y}\right) - \lambda(1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1}$$

$$= (1+y)^{-1} s_{\lambda}(y) - \lambda(1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \int_o^y \left(\frac{t}{1+t}\right)^n \frac{dt}{(1+t)^2} \ .$$

Now, as

$$\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} \frac{|s_n|}{n+1} \left(\frac{y}{1+y}\right)^{n+1} < \infty ,$$

we may invert the order of summation and integration and obtain

$$u_{\lambda}(y) = (1+y)^{-1} s_{\lambda}(y) - (1+y)^{-\lambda-1} \int_o^y \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n \left(\frac{t}{1+t}\right)^n \frac{dt}{(1+t)^2}$$

$$= (1+y)^{-1} s_{\lambda}(y) - \lambda(1+y)^{-\lambda-1} \int_o^y (1+t)^{\lambda-1} s_{\lambda}(t)dt.$$

This establishes (2.2.1).

Integrating by parts on the right hand side of (2.2.1) we get

$$u_{\lambda}(y) = (1+y)^{-1} s_{\lambda}(y) - (1+y)^{-\lambda-1}\left[(1+y)^{\lambda} s_{\lambda}(y) - s_{\lambda}(0) - \int_o^y (1+t)^{\lambda} s_{\lambda}'(t)dt\right]$$

$$= s_\lambda(0)\ (1+y)^{-\lambda-1} + (1+y)^{-\lambda-1}\int_0^y (1+t)^\lambda s_\lambda'(t)dt,$$

which is (2.2.2).

Next, integrating both sides in (2.2.1) we have

$$\int_0^y u_\lambda(x)dx = \int_0^y (1+x)^{-1}s_\lambda(x)dx - \lambda\int_0^y (1+x)^{-\lambda-1}dx\int_0^x (1+t)^{\lambda-1}s_\lambda(t)dt$$

$$= \int_0^y (1+x)^{-1}s_\lambda(x)dx - \int_0^y (1+t)^{\lambda-1}s_\lambda(t)dt\int_t^y \lambda(1+x)^{-\lambda-1}dx$$

$$= \int_0^y (1+x)^{-1}s_\lambda(x)dx - \int_0^y (1+t)^{\lambda-1}s_\lambda(t)[(1+t)^{-\lambda} - (1+y)^{-\lambda}]dt$$

$$= (1+y)^{-\lambda}\int_0^y (1+t)^{\lambda-1}s_\lambda(t)dt,$$

and (2.2.3) follows.

Combining (2.2.1) and (2.2.3) we get

$$s_\lambda(y) = \lambda(1+y)^{-\lambda}\int_0^y (1+t)^{\lambda-1}s_\lambda(t)dt + (1+y)u_\lambda(y)$$

$$= U_\lambda(y) + (1+y)u_\lambda(y).$$

This is (2.2.4).

By (1.3.2) and (2.2.4) we have

$$s_\lambda(y) = (\lambda+1)y^{-\lambda-1}\int_0^y t^\lambda s_{\lambda+1}(t)dt$$

$$= (\lambda+1)y^{-\lambda-1}\int_0^y t^\lambda[U_{\lambda+1}(t) + (1+t)u_{\lambda+1}(t)]dt$$

$$= (\lambda+1)y^{-\lambda-1}\int_0^y t^{\lambda}U_{\lambda+1}(t)dt + (\lambda+1)y^{-\lambda-1}\int_0^y t^{\lambda}u_{\lambda+1}(t)dt$$

$$+ (\lambda+1)y^{-\lambda-1} \int_0^y t^{\lambda+1}u_{\lambda+1}(t)dt.$$

Integration by parts yields

$$s_{\lambda}(y) = (\lambda+1)y^{-\lambda-1}\int_0^y t^{\lambda}U_{\lambda+1}(t)dt + (\lambda+1)y^{-\lambda-1}\int_0^y t^{\lambda}u_{\lambda+1}(t)dt$$

$$+ y^{-\lambda-1}[t^{\lambda+1}U_{\lambda+1}(t)\Big|_0^y - (\lambda+1)\int_0^y t^{\lambda}U_{\lambda+1}(t)dt]$$

$$= U_{\lambda+1}(y) + (\lambda+1)y^{-\lambda-1}\int_0^y t^{\lambda}u_{\lambda+1}(t)dt.$$

Using (1.3.2) with u in place of s, we obtain (2.2.5).

In view of (2.2.4) and (2.2.5) we have

$$yu_{\lambda}(y) = U_{\lambda+1}(y) - U_{\lambda}(y),$$

which is (2.2.6). We will also put it in the following form

$$(2.2.6') \qquad y\frac{d}{dy}[U_{\lambda}(y)] = \frac{1}{\lambda}[U_{\lambda+1}(y) - U_{\lambda}(y)].$$

For the last part of the lemma we use (1.3.2) with u in place of s to obtain

$$u_{\lambda+1}(y) = (\lambda+2)y^{-\lambda-2}\int_0^y t^{\lambda+1}u_{\lambda+2}(t)dt.$$

Integrating partially we obtain

$$u_{\lambda+1}(y) = y^{-\lambda-2}[y^{\lambda+1}U_{\lambda+2}(y) - (\lambda+1)\int_o^y t^\lambda U_{\lambda+2}(t)dt]$$

$$= y^{-1}U_{\lambda+2}(y) - (\lambda+1)y^{-\lambda-2}\int_o^y t^\lambda U_{\lambda+2}(t)dt,$$

which together with (2.2.6) yields

$$U_{\lambda+1}(y) = (\lambda+1)y^{-\lambda-1}\int_o^y t^\lambda U_{\lambda+2}(t)dt.$$

This completes the proof of the lemma.

The next lemma is basic for the proof of Theorem 2.4. The analogous result for the case of ordinary Abel-type summability has been proved by Borwein [4] (Lemma 4).

LEMMA 2.2

If $\lambda > -1$, a *is real, and if*

(2.2.1) $$s_n \to \ell \ (A_\lambda)$$

and

(2.2.2) $$(n+a)v_n = s_n \quad (n = 0, 1, 2, \ldots)$$

then

(2.2.3) $$v_n \to 0 \ |A_\lambda|.$$

PROOF.

Let

(2.2.4) $$\phi(x) = \sum_{n=m}^{\infty} \varepsilon_n^\lambda s_n x^{n+a-1} \quad (|x| < 1)$$

where $m > |a| + 1$.

By (2.2.1) we have, for $0 \leq x < 1$,

(2.2.5)
$$|\phi(x)| < M(1-x)^{-\lambda-1} ,$$

where M is a positive constant.

Let

(2.2.6)
$$\psi(x) = (1-x)^{\lambda+1} \sum_{n=m}^{\infty} \varepsilon_n^{\lambda} v_n x^n .$$

Since the series defining $\psi(x)$ is convergent in (0, 1) and a polynomial function is of bounded variation in any finite interval, it suffices to show, after Borwein's Lemma 4 in [4], for the proof of our lemma that $\psi(x)$ is of bounded variation in $[\frac{1}{2}, 1)$.

Now, in view of (2.2.2) we have

(2.2.7)
$$x^a \psi(x) = (1-x)^{\lambda+1} \int_0^x \phi(t)dt.$$

Evidently it is enough to show that $x^a \psi(x)$ is of bounded variation in $(\frac{1}{2}, 1)$.

By (2.2.6) we have

$$\int_{1/2}^{1} \left|\frac{d}{dx}(x^a \psi(x))\right| dx = \int_{1/2}^{1} \left|-(\lambda+1)(1-x)^{\lambda} \int_0^x \phi(t)dt + (1-x)^{\lambda+1}\phi(x)\right| dx .$$

$$\leq \int_{1/2}^{1}(\lambda+1)(1-x)^{\lambda}dx\int_{o}^{x}|\phi(t)|dt + \int_{1/2}^{1}(1-x)^{\lambda+1}|\phi(x)|dx$$

$$\leq \int_{o}^{1}(\lambda+1)(1-x)^{\lambda}dx\int_{o}^{x}|\phi(t)|dt + \int_{o}^{1}(1-x)^{\lambda+1}|\phi(x)|dx.$$

Taking note of (2.2.5), we get

$$\int_{1/2}^{1}\left|\frac{d}{dx}\,(x^{a}\psi(x))\right|dx \leq M\int_{o}^{1}(\lambda+1)(1-x)^{\lambda}dx\,\int_{o}^{x}(1-t)^{-\lambda-1}dt + M$$

$$= M\int_{o}^{1}(1-t)^{-\lambda-1}dt\,\int_{t}^{1}(\lambda+1)(1-x)^{\lambda}dx + M$$

$$= M < \infty.$$

The lemma follows.

An immediate consequence of the above lemma is the following.

LEMMA 2.3

If $\lambda > -1$, p *and* q *are real and* $s_n \to \ell\ |A_\lambda|$, *then*

$$\frac{n+p}{n+q}\,s_n \to \ell\ |A_\lambda|.$$

PROOF.

By Lemma 2.2, we have

$$\frac{n+p}{n+q}\,s_n = s_n + \frac{p-q}{n+q}\,s_n \to \ell|A_\lambda|.$$

REMARK

It might be noted here that Lemma 2.2 has, in fact been established under the weaker hypothesis $s_n = O(1)(A_\lambda)$ than (2.2.1).

2.3

With the results of §2.2 at our disposal, we will establish the following theorems.

THEOREM 2.1

The method $|A_\lambda|$ *is translative for* $\lambda > -1$.

PROOF.

Suppose that $s_n \to \ell\ |A_\lambda|$.

Since

$$\varepsilon_{n+1}^{\lambda} = \left(1 + \frac{\lambda}{n+1}\right)\varepsilon_n^{\lambda} ,$$

we have the following identity, for $0 \leq x < 1$,

$$(2.3.1) \quad (1-x)^{\lambda+1} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda} s_{n-1} x^n = (1-x)^{\lambda+1} x \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n x^n + \lambda(1-x)^{\lambda+1} x \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} \frac{s_n}{n+1} x^n .$$

Applying Lemma 2.2, it follows that the right hand side of (2.3.1) is of bounded variation in [0, 1).

Hence, taking note of the translativity of A_λ (Borwein [4] Theorem 5) we have that

$$s_{n-1} \to \ell\ |A_\lambda|.$$

Also, since

$$\varepsilon_{n-1}^{\lambda} = (1 - \frac{\lambda}{n+\lambda})\varepsilon_n^{\lambda} ,$$

we have, for $0 \leq x < 1$,

$$(2.3.2) \quad (1-x)^{\lambda+1} x \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_{n+1} x^n$$

$$= (1-x)^{\lambda+1} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda} s_n x^n - \lambda(1-x)^{\lambda+1} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda} \frac{s_n}{n+\lambda} x^n .$$

Now, $(1-x)^{\lambda+1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_{n+1} x^n$ is of bounded variation in $[0, \frac{1}{2}]$ by the convergence of the series in $[0, 1)$. By our assumption ($s_n \to \ell|A_\lambda|$) and Lemma 2.2, the right hand side of (2.3.2) is of bounded variation in $[\frac{1}{2}, 1)$. Since $\frac{1}{x}$ is of bounded variation in $[\frac{1}{2}, 1)$, it follows that $(1-x)^{\lambda+1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_{n+1} x^n$ is of bounded variation in $[\frac{1}{2}, 1)$.

Combining these results and taking note of the translativity of A_λ, we have that

$$s_{n+1} \to \ell \ |A_\lambda| .$$

This completes the proof of the theorem.

The next theorem provides the analogue of Theorem 1.3 for $|A'_\lambda|$-convergence.

THEOREM 2.2

If $\lambda > \mu > 0$ *and* $s_n \to \ell\ |A'_\lambda|$, *then* $s_n \to \ell\ |A'_\mu|$.

PROOF.

By Theorem 1.3 and Theorem B, we have that

$$s_n \to \ell\ (A'_\mu) \quad \text{whenever} \quad s_n \to \ell\ (A'_\lambda). \tag{2.3.3}$$

Hence, it remains to prove that $U_\mu(y)$ is of bounded variation in $[0, \infty)$ whenever $U_\lambda(y)$ is of bounded variation in $[0, \infty)$. By (1.3.2) with u in place of s, we have that

$$|u_\mu(y)| \le \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)}\, y^{-\lambda}\int_0^y (y-t)^{\lambda-\mu-1}\, t^\mu |u_\lambda(t)|\,dt$$

Thus,

$$\int_0^\infty |u_\mu(y)|\,dy \le \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} \int_0^\infty y^{-\lambda}dy \int_0^y (y-t)^{\lambda-\mu-1}\, t^\mu |u_\lambda(t)|\,dt$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} \int_0^\infty t^\mu |u_\lambda(t)|\,dt \int_t^\infty (y-t)^{\lambda-\mu-1} y^{-\lambda}dy\ .$$

It is easily verified that

$$\int_t^\infty (y-t)^{\lambda-\mu-1}\, y^{-\lambda}dy = t^{-\mu}\, \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\ .$$

Hence,

$$\int_0^\infty |u_\mu(y)|\,dy \le \frac{\lambda}{\mu} \int_0^\infty |u_\lambda(t)|\,dt$$

$$< \infty \quad ,$$

i.e., $U_\mu(y)$ is of bounded variation in $[0, \infty)$. It follows that $U_\mu(y) \to \ell'$ (say) as $y \to \infty$. That $\ell' = \ell$ is a consequence of (2.3.3).

We next obtain a sufficient condition for $|A_\lambda'|$- convergence.

THEOREM 2.3

If $\lambda > 0$ *and*

$$\int_0^\infty y^{-1}|U_{\lambda+1}(y) - \ell|dy < \infty \tag{2.3.4}$$

then $s_n \to \ell \; |A_\lambda'|$.

PROOF.

We have, by (2.2.7),

$$U_\lambda(y) - \ell = \lambda y^{-\lambda}\int_0^y t^{\lambda-1}\, U_{\lambda+1}(t)dt - \ell$$

$$= \lambda y^{-\lambda}\int_0^y t^{\lambda-1}(U_{\lambda+1}(t) - \ell)dt;$$

and so

$$\int_0^\infty y^{-1}|U_\lambda(y) - \ell|dy \le \lambda\int_0^\infty y^{-\lambda-1}\, dy \int_0^y t^{\lambda-1}|U_{\lambda+1}(t) - \ell|dt$$

$$= \int_0^\infty t^{\lambda-1}|U_{\lambda+1}(t) - \ell|dt \int_t^\infty \lambda y^{-\lambda-1}dy$$

$$= \int_0^\infty t^{-1}|U_{\lambda+1}(t) - \ell|dt.$$

Thus, by (2.3.4) it follows that

$$\int_0^\infty y^{-1}|U_\lambda(y) - \ell|\,dy < \infty . \tag{2.3.5}$$

Now, by (2.2.6) we have that

$$\begin{aligned}\int_0^\infty |u_\lambda(y)|\,dy &= \int_0^\infty y^{-1}|U_{\lambda+1}(y) - U_\lambda(y)|\,dy \\ &\leq \int_0^\infty y^{-1}|U_{\lambda+1}(y) - \ell|\,dy + \int_0^\infty y^{-1}|U_\lambda(y) - \ell|\,dy \\ &\leq 2\int_0^\infty y^{-1}|U_{\lambda+1}(y) - \ell|\,dy \\ &< \infty \quad ,\end{aligned}$$

i.e., $U_\lambda(y)$ is of bounded variation in $[0, \infty)$.
It follows that the sequence $\{s_n\}$ is $|A'_\lambda|$-convergent to ℓ' (say).
That $\ell' = \ell$ is a consequence of (2.3.5).
This completes the proof of the theorem.
The next two theorems are the absolute summability analogues of Theorems A and B respectively.

THEOREM 2.4

For $\lambda > 0$, $s_n \to \ell\ |A_\lambda|$ *if and only if* $s_n \to \ell\ |A'_\lambda|$ *and* $nu_n \to 0\ |A_{\lambda-1}|$.

PROOF.

(i) Suppose that $s_n \to \ell\ |A_\lambda|$, i.e., $s_\lambda(y)$ is of bounded variation in $[0, \infty)$.

By (2.2.2), we have that

$$|u_\lambda(y)| \leq |s_\lambda(0)|(1+y)^{-\lambda-1} + (1+y)^{-\lambda-1}\int_0^y (1+t)^\lambda |s_\lambda'(t)|dt,$$

and so

$$\int_0^\infty |u_\lambda(y)|dy \leq |s_\lambda(0)| \int_0^\infty (1+y)^{-\lambda-1}\, dy$$

$$+ \int_0^\infty (1+y)^{-\lambda-1}\, dy \int_0^y (1+t)^\lambda |s_\lambda'(t)|dt$$

$$= \frac{1}{\lambda}|s_\lambda(0)| + \int_0^\infty (1+t)^\lambda |s_\lambda'(t)|dt \int_t^\infty (1+y)^{-\lambda-1}dy$$

$$= \frac{1}{\lambda}|s_\lambda(0)| + \frac{1}{\lambda}\int_0^\infty |s_\lambda'(t)|dt$$

$$< \infty \quad ,$$

i.e., $U_\lambda(y)$ is of bounded variation in $[0, \infty)$ and hence tends to ℓ' (say) as $y \to \infty$.

It follows, by Theorem A, that $\ell' = \ell$ and so

$$s_n \to \ell \ |A_\lambda'|.$$

Further, by (2.2.4), we have that $(1+y)u_\lambda(y)$ is of bounded variation in $[0, \infty)$.

But

$$(1+y)u_\lambda(y) = (1+y)^{-\lambda} \sum_{n=0}^\infty \varepsilon_n^\lambda u_n \left(\frac{y}{1+y}\right)^n .$$

Since

$$\varepsilon_n^{\lambda} = \frac{\lambda+n}{\lambda} \quad \varepsilon_n^{\lambda-1} \quad ,$$

we have that

$$(2.3.6) \quad (1+y)u_\lambda(y) = (1+y)^{-\lambda} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda-1} \frac{\lambda+n}{\lambda n} nu_n\left(\frac{y}{1+y}\right)^n + u_o(1+y)^{-\lambda}.$$

Now $(1+y)^{-\lambda}$ is of bounded variation in $[0, \infty)$ and tends to 0 as $y \to \infty$.

Hence

$$(1+y)^{-\lambda} \sum_{n=1}^{\infty} \varepsilon_n^{\lambda-1} \frac{\lambda+n}{\lambda n} \quad nu_n\left(\frac{y}{1+y}\right)^n$$

is of bounded variation in $[0, \infty)$. It follows from (2.2.4) and (2.3.6) that

$$\frac{\lambda+n}{\lambda n} nu_n \to 0 \;|A_{\lambda-1}|.$$

Consequently, by Lemma 2.3 ,

$$nu_n \to 0 \;|A_{\lambda-1}|.$$

This completes the proof of (i).

(ii) Suppose that

$$(2.3.7) \qquad s_n \to \ell \;|A'_\lambda| \quad \text{and}$$

$$(2.3.8) \qquad nu_n \to 0 \;|A_{\lambda-1}|.$$

By (2.3.8) and Lemma 2.3 ,

$$\frac{\lambda+n}{\lambda n} \cdot nu_n \to 0 \; |A_{\lambda-1}|,$$

and so, by (2.3.6), $(1+y)u_\lambda(y)$ is of bounded variation in $[0, \infty)$. Thus, by (2.2.4) and (2.3.7),

$$s_n \to \ell \; |A_\lambda|.$$

This completes the proof of the theorem.

THEOREM 2.5

For $\lambda > 0$, $|A'_\lambda| \approx |A_{\lambda-1}|$.

PROOF.

(i) Suppose that $s_n \to \ell \; |A'_\lambda|$, i.e., $U_\lambda(y)$ is of bounded variation in $[0, \infty)$.

Now, by (2.2.5),

(2.3.9) $$s_{\lambda-1}(y) = U_\lambda(y) + u_{\lambda-1}(y)$$

and

$$u_{\lambda-1}(y) = (1+y)^{-\lambda} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda-1} \; u_n \left(\frac{y}{1+y}\right)^n .$$

Since the series defining $u_{\lambda-1}(y)$ converges for $y > 0$, and $(1+y)^{-\lambda}$ is of bounded variation in $[0, \infty)$, it follows that $u_{\lambda-1}(y)$ is of bounded variation in $[0, 1]$.

In virtue of (2.3.9) and Theorem B, it suffices to show that $u_{\lambda-1}(y)$ is of bounded variation in $[1, \infty)$.

Now by (1.3.2) with u in place of s, we have that

$$u_{\lambda-1}(y) = \lambda y^{-\lambda}\int_0^y t^{\lambda-1} u_\lambda(t)dt,$$

and so

$$\int_1^\infty \left|\frac{d}{dy} u_{\lambda-1}(y)\right|dy \leq \lambda^2 \int_1^\infty y^{-\lambda-1}dy \int_0^y t^{\lambda-1}|u_\lambda(t)|dt + \lambda\int_1^\infty y^{-1}|u_\lambda(y)|dy$$

$$\leq \lambda\int_0^1 t^{\lambda-1}|u_\lambda(t)|dt \int_1^\infty \lambda y^{-\lambda-1}dy$$

$$+\lambda\int_1^\infty t^{\lambda-1}|u_\lambda(t)|dt \int_t^\infty \lambda\, y^{-\lambda-1}dy + \lambda\int_0^\infty |u_\lambda(y)|dy.$$

$$\leq \lambda\int_0^1 t^{\lambda-1}|u_\lambda(t)|dt + 2\lambda\int_0^\infty |u_\lambda(y)|dy$$

$$< \infty \qquad .$$

Hence

$$s_n \to \ell\ |A_{\lambda-1}|.$$

(ii) Suppose that $s_n \to \ell|A_{\lambda-1}|$ i.e., $s_{\lambda-1}(y)$ is of bounded variation in $[0, \infty)$.

In view of (2.3.9) and Theorem B, it suffices to show that $u_{\lambda-1}(y)$ is of bounded variation in $[0, \infty)$.

Now, by (2.2.2) we have that

$$\int_0^\infty |u'_{\lambda-1}(y)|dy$$

$$= \int_0^\infty \left|\frac{d}{dy}\left\{(1+y)^{-\lambda} s_{\lambda-1}(0) + (1+y)^{-\lambda}\int_0^y (1+t)^{\lambda-1} s'_{\lambda-1}(t)dt\right|dy\right.$$

$$\leq |s_{\lambda-1}(0)| + \int_0^\infty \lambda(1+y)^{-\lambda-1}\, dy \int_0^y (1+t)^{\lambda-1}\, |s'_{\lambda-1}(t)|dt$$

$$+ \int_0^\infty (1+y)^{-1}|s'_{\lambda-1}(y)|dt$$

$$\leq |s_{\lambda-1}(0)| + \int_0^\infty (1+t)^{\lambda-1}|s'_{\lambda-1}(t)|dt \int_t^\infty \lambda(1+y)^{-\lambda-1}dy$$

$$+ \int_0^\infty |s'_{\lambda-1}(y)|dy$$

$$\leq |s_{\lambda-1}(0)| + 2\int_0^\infty |s'_{\lambda-1}(y)|dy$$

$$< \infty \quad .$$

Hence

$$s_n \to \ell \; |A'_\lambda| .$$

The proof of the theorem is now complete.

The following theorems are the immediate consequence of Theorem 2.5 and Theorems 2.1 and 2.2 respectively.

THEOREM 2.6

The method $|A'_\lambda|$ *is translative for* $\lambda > 0$.

THEOREM 2.7

If $\lambda > \mu > -1$ *and* $s_n \to \ell \; |A_\lambda|$, *then* $s_n \to \ell \; |A_\mu|$.

CHAPTER 3

STRONG ABEL-TYPE SUMMABILITY - I

In this chapter strong summability methods based upon the Abel-type methods A_λ and A'_λ are considered and some of their properties investigated.

The notion of strong summability was first introduced in 1916 by M. Fekete [11], who defined summability [C, 1] - the strong Cesàro summability of order 1. In 1933, C.E. Winn [34] extended the scope of strong summability to summability [C, k] - the strong Cesàro summability of positive order k. Later on, the concept of strong summability has been applied to other methods of summability by various authors. A definition for strong summability for a general class of matrix summability methods is given by Borwein [10]. P. Srivastava [29] and [30] gives definitions for strong summability for sequence to function methods.

The strong Abel summability was first considered by Harrington and Hyslop [16]. Two other definitions of strong Abel-summability were given by Flett [13]. The strong Abel-type summability $[A_\lambda]$ has been investigated by Mishra [22].

3.1 *DEFINITIONS*

Let $s_\lambda(y)$, $u_\lambda(y)$ and $U_\lambda(y)$ be as defined in §1.2.

Strong Abel-type summability $[A_\lambda]$.

If

$$\int_0^y |s_{\lambda+1}(t) - \ell|\,dt = o(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A_λ*-convergent* or $[A_\lambda]$*-convergent* to ℓ and write

$$s_n \to \ell\ [A_\lambda].$$

Strong Abel-type summability $[A'_\lambda]$.

If

$$\int_0^y |U_{\lambda+1}(t) - \ell|\,dt = o(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A'_λ*-convergent* or $[A'_\lambda]$*-convergent* to ℓ and write

$$s_n \to \ell\ [A'_\lambda].$$

Strong boundedness.

If

$$\int_0^y |s_{\lambda+1}(t)|\,dt = O(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A_λ*-bounded* or $[A_\lambda]$*-bounded* and write

$$s_n = O(1)\ [A_\lambda]\ .$$

Strong $[A'_\lambda]$-boundedness is defined analogously.

3.2

This section contains some consequences of the definitions and two lemmas which are used in the following section.

THEOREM 3.1

If $\lambda > 0$ *and* $s_n \to \ell\ [A'_\lambda]$, *then*

$$s_n \to \ell\ (A'_\lambda)\ . \tag{3.2.1}$$

PROOF.

We have, by (2.2.7), that

$$U_\lambda(y) - \ell = \lambda y^{-\lambda} \int_0^y t^{\lambda-1}(U_{\lambda+1}(t) - \ell)dt.$$

Hence,

$$|U_\lambda(y) - \ell| \le \lambda y^{-\lambda} \int_0^y t^{\lambda-1}|U_{\lambda+1}(t) - \ell|dt,$$

and so, integrating by parts, we have

$$|U_\lambda(y) - \ell| \leq \lambda y^{-\lambda}\left[t^{\lambda-1}o(t)\Big|_o^y - (\lambda-1)\int_o^y t^{\lambda-2}\, o(t)dt\right]$$

$$= o(1) \quad \text{as} \quad y \to \infty .$$

i.e.

$$s_n \to \ell \ (A'_\lambda) .$$

This proves Theorem 3.1.

An immediate consequence is the following

COROLLARY 3.1.1

If $\lambda > 0$ *and* $s_n = O(1)\ [A'_\lambda]$, *then*

$$s_n = O(1)\ (A'_\lambda) .$$

THEOREM 3.2

If $\lambda > 0$ *and* $s_n \to \ell\ (A'_\lambda)$, *then*

(3.2.2) $$s_n \to \ell\ [A'_{\lambda-1}]$$

PROOF.

By hypothesis, we have that

(3.2.3) $$U_\lambda(t) - \ell = o(1) \quad \text{as} \quad t \to \infty.$$

Next, we observe that:

(a) The strong Abel-type convergence $[A'_\lambda]$ of the sequence $\{s_n\}$ to ℓ is equivalent to the (C, 1)-summability of the function $|U_{\lambda+1}(t) - \ell|$ to 0.

(b) The Cesàro method (C, 1) is regular.

The theorem follows as a consequence of (3.2.3), (a) and (b).

COROLLARY 3.2.1

If $\lambda > 0$ *and* $s_n = O(1)\ (A'_\lambda)$, *then*

$$s_n = O(1)\ [A'_{\lambda-1}].$$

We note the following consequence of Theorem 3.1 and Theorem 3.2.

THEOREM 3.3

If $\lambda > 0$ *and* $s_n \to \ell\ [A'_\lambda]$, *then*

$$s_n \to \ell\ [A'_{\lambda-1}]$$

COROLLARY 3.3.1

If $\lambda > 0$ *and* $s_n = O(1)\ [A'_\lambda]$, *then* $s_n = O(1)\ [A'_{\lambda-1}]$.

The following theorem gives the necessary and sufficient conditions for $[A'_\lambda]$-convergence.

THEOREM 3.4

For $\lambda > 0$, *the necessary and sufficient conditions for the* $[A'_\lambda]$-*convergence of the sequence* $\{s_n\}$ *to* ℓ *are that*

$$s_n \to \ell\ (A'_\lambda) \tag{3.2.4}$$

and

$$\int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right| dt = o(y), \text{ as } y \to \infty. \tag{3.2.5}$$

PROOF.

Necessity:

Suppose that $s_n \to \ell\ [A_\lambda']$.

Then (3.2.4) follows from Theorem 3.1.

Now, by (2.2.6') (Page 15), we have that

$$\int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right| dt = \frac{1}{\lambda} \int_0^y |U_{\lambda+1}(t) - U_\lambda(t)| dt$$

$$\leq \frac{1}{\lambda} \int_0^y |U_{\lambda+1}(t) - \ell| dt + \frac{1}{\lambda} \int_0^y |U_\lambda(t) - \ell| dt$$

$$= o(y),$$

by Theorem 3.3.

This completes the proof of the necessity part.

Sufficiency:

Suppose that (3.2.4) and (3.2.5) hold.

Now, by (3.2.4) and Theorem 3.2, we have that

$$\int_0^y |U_\lambda(t) - \ell| dt = o(y).$$

Hence, it follows, by (2.2.6') and (3.2.5) that

$$\int_0^y |U_{\lambda+1}(t) - \ell| dt \leq \lambda \int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right| dt + \int_0^y |U_\lambda(t) - \ell| dt$$

$$= o(y).$$

The theorem follows.

COROLLARY 3.4.1

For $\lambda > 0$, $s_n = O(1)\ [A'_\lambda]$ *if and only if*

(3.2.6) $$s_n = O(1)\ (A'_\lambda)$$

and

(3.2.7) $$\int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right| dt = O(y) \quad \text{as } y \to \infty .$$

The following theorem is due to Mishra [21] (Theorem 6). For the sake of completeness the proof is given here.

THEOREM 3.5

If $\lambda > -1$ *and*

(3.2.8) $$s_n \to \ell\ |A_\lambda|,$$

then

(3.2.9) $$s_n \to \ell\ [A_\lambda].$$

PROOF.

We have, by (1.3.1), that

$$s'_\lambda(z) = -\frac{(\lambda+1)}{z}\,[s_{\lambda+1}(z) - s_\lambda(z)]$$

$$= \frac{1}{z}\, t_\lambda(z),$$

and so, by (3.2.8), we have that

(3.2.10) $$\int_0^y |s'_\lambda(z)|dz = \int_0^y \frac{1}{z}|t_\lambda(z)|dz \to M < \infty, \quad \text{as } y \to \infty .$$

Since

$$\frac{1}{y}\int_0^y |t_\lambda(z)|dz = \int_0^y \frac{1}{z}|t_\lambda(z)|dz - \frac{1}{y}\int_0^y (y-z)\,\frac{1}{z}\,|t_\lambda(z)|dz,$$

it follows from (3.2.10) and the regularity of the (C, 1)-method, that

(3.2.11) $$\frac{1}{y}\int_0^y |t_\lambda(z)|dz = o(1), \quad \text{as } y \to \infty .$$

Now, we have by (1.2.1) that

$$\frac{1}{y}\int_0^y |s_{\lambda+1}(z) - \ell|dz = \frac{1}{y}\int_0^y |s_\lambda(z) - \ell + \frac{1}{\lambda+1}\,t_\lambda(z)|dz$$

$$\leq \frac{1}{y}\int_0^y |s_\lambda(z) - \ell|dz + \frac{1}{\lambda+1}\,\frac{1}{y}\int_0^y |t_\lambda(z)|dz .$$

Since $s_n \to \ell\ |A_\lambda|$ implies that $s_\lambda(z) - \ell = o(1)$ as $z \to \infty$, we have that

(3.2.12) $$\frac{1}{y}\int_0^y |s_\lambda(z) - \ell|dz = o(1)$$

It now follows from (3.2.11) and (3.2.12) that,

$$\frac{1}{y}\int_0^y |s_{\lambda+1}(z) - \ell|dz = o(1), \quad \text{as } y \to \infty .$$

The theorem follows.

LEMMA 3.1

If $\lambda > -1$, *a is real, and if*

$$s_n \to \ell\ (A_\lambda)$$

and

$$(n+a)v_n = s_n \qquad (n = 0, 1, 2, \ldots),$$

then

$$v_n \to 0\ [A_\lambda].$$

PROOF.

We have, by Lemma 2.2, that

$$v_n \to 0\ |A_\lambda|.$$

Hence, by Theorem 3.5, it follows that

$$v_n \to 0\ [A_\lambda].$$

This completes the proof of the lemma.

LEMMA 3.2

If $\lambda > -1$, p *and* q *are real and*

$$s_n \to \ell\ [A_\lambda],$$

then

$$\frac{n+p}{n+q}\, s_n \to \ell\ [A_\lambda].$$

PROOF.

Since $[A_\lambda] \Rightarrow (A_\lambda)$, the lemma is a consequence of Lemma 3.1.

3.3

In this section we prove the strong summability analogues of Theorems A and B (Chapter 1).

THEOREM 3.6

For $\lambda > 0$, $s_n \to \ell\ [A_\lambda]$ *if and only if* $s_n \to \ell\ [A'_\lambda]$ *and* $nu_n \to 0\ [A_{\lambda-1}]$.

PROOF.

(i) Suppose that $s_n \to \ell\ [A_\lambda]$, i.e.

$$(3.3.1) \qquad \int_0^y |s_{\lambda+1}(t) - \ell|dt = o(y), \text{ as } y \to \infty .$$

Now, by (2.2.3), we have that

$$U_{\lambda+1}(t) - \ell = (\lambda+1)(1+t)^{-\lambda-1} \int_0^t (1+z)^\lambda\, s_{\lambda+1}(z)dz - \ell$$

$$= (\lambda+1)(1+t)^{-\lambda-1} \int_0^t (1+z)^\lambda [s_{\lambda+1}(z) - \ell]dz - \ell(1+t)^{-\lambda-1} ,$$

and so

$$(3.3.2) \quad \int_0^y |U_{\lambda+1}(t) - \ell|dt$$

$$\leq (\lambda+1) \int_0^y (1+t)^{-\lambda-1}dt \int_0^t (1+z)^\lambda |s_{\lambda+1}(z) - \ell|dz$$

$$+ |\ell| \int_0^y (1+t)^{-\lambda-1}dt.$$

But

$$(3.3.3) \quad \int_0^y (1+t)^{-\lambda-1}dt = \frac{1}{\lambda}[1 - (1+y)^{-\lambda}]$$

$$= O(1) = o(y) \text{ as } y \to \infty ,$$

and

$$\int_o^y (1+t)^{-\lambda-1}\, dt \int_o^t (1+z)^\lambda |s_{\lambda+1}(z) - \ell|\, dz$$

$$= \int_o^y (1+z)^\lambda |s_{\lambda+1}(z) - \ell|\, dz \int_z^y (1+t)^{-\lambda-1} dt$$

$$= \int_o^y (1+z)^\lambda |s_{\lambda+1}(z) - \ell|\, dz \left\{ \frac{1}{\lambda} [(1+z)^{-\lambda} - (1+y)^{-\lambda}] \right\}$$

$$= \frac{1}{\lambda} \int_o^y |s_{\lambda+1}(z) - \ell|\, dz - \frac{1}{\lambda} (1+y)^{-\lambda} \int_o^y (1+z)^\lambda |s_{\lambda+1}(z) - \ell|\, dz.$$

$$= o(y) - \frac{1}{\lambda} I(y),$$

where

$$I(y) = (1+y)^{-\lambda} \int_o^y (1+z)^\lambda |s_{\lambda+1}(z) - \ell|\, dz$$

$$\leq \int_o^y |s_{\lambda+1}(z) - \ell|\, dz$$

$$= o(y) \quad \text{as} \quad y \to \infty.$$

Hence, it follows from (3.3.2) that

$$(3.3.4) \qquad \int_o^y |U_{\lambda+1}(t) - \ell|\, dt = o(y), \quad \text{as } y \to \infty ,$$

i.e.,

$$s_n \to \ell \ [A_\lambda'].$$

Further, by (2.2.4), (Cf. Chap. 2, page 25)

$$(3.3.5)\quad (1+t)^{-\lambda-1}\sum_{n=1}^{\infty}\varepsilon_n^{\lambda}\frac{\lambda+1+n}{(\lambda+1)n}\,nu_n\left(\frac{t}{1+t}\right)^n = s_{\lambda+1}(t) - U_{\lambda+1}(t) - u_o(1+t)^{-\lambda-1}$$

and so

$$\int_o^y \left|(1+t)^{-\lambda-1}\sum_{n=1}^{\infty}\varepsilon_n^{\lambda}\frac{\lambda+1+n}{(\lambda+1)n}\,nu_n\left(\frac{t}{1+t}\right)^n\right|dt$$

$$\leq \int_o^y |s_{\lambda+1}(t) - \ell|\,dt + \int_o^y |U_{\lambda+1}(t) - \ell|\,dt + |u_o|\int_o^y (1+t)^{-\lambda-1}dt$$

It follows, from (3.3.1), (3.3.3) and (3.3.4), that

$$(3.3.6)\quad \int_o^y \left|(1+t)^{-\lambda-1}\sum_{n=1}^{\infty}\varepsilon_n^{\lambda}\frac{\lambda+1+n}{(\lambda+1)n}\,nu_n\left(\frac{t}{1+t}\right)^n\right| dt = o(y),\text{ as } y\to\infty;$$

i.e.

$$\frac{\lambda+1+n}{(\lambda+1)n}\,nu_n \to 0\ [A_{\lambda-1}].$$

Consequently, by Lemma 3.2,

$$nu_n \to 0\ [A_{\lambda-1}].$$

This completes the proof of (i).

(ii) Suppose that $s_n \to \ell\ [A'_\lambda]$ i.e. (3.3.4) holds and that $nu_n \to 0\ [A_{\lambda-1}]$.

It follows from Lemma 3.2, that

$$\frac{\lambda+1+n}{(\lambda+1)n}\, nu_n \to 0\ [A_{\lambda-1}]$$

i.e. (3.3.6) holds.

Hence, by (3.3.5), (3.3.3) and (3.3.4), we have that

$$\int_0^y |s_{\lambda+1}(t) - \ell|\,dt = o(y), \text{ as } y \to \infty\ ;$$

i.e.

$$s_n \to \ell\ [A_\lambda].$$

The theorem is proved.

THEOREM 3.7

For $\lambda > 0$, $[A'_\lambda] \simeq [A_{\lambda-1}]$.

PROOF.

(i) Suppose that $s_n \to \ell\ [A'_\lambda]$ i.e.,

(3.3.7) $$\int_0^y |U_{\lambda+1}(t) - \ell|\,dt = o(y), \text{ as } y \to \infty\ .$$

By (2.2.5), it follows that

$$\int_0^y |s_\lambda(t) - \ell|\,dt \le \int_0^y |U_{\lambda+1}(t) - \ell|\,dt + \int_0^y |u_\lambda(t)|\,dt$$

$$= o(y) + I(y),$$

by (3.3.7), where

$$I(y) = \int_o^y |u_\lambda(t)|dt = \int_o^1 |u_\lambda(t)dt + \int_1^y |u_\lambda(t)|dt$$

$$= o(y) + \int_1^y |u_\lambda(t)|dt.$$

Now, by (2.2.6), we have

$$\int_1^y |u_\lambda(t)|dt = \int_1^y \frac{1}{t} |U_{\lambda+1}(t) - U_\lambda(t)|dt$$

$$\leq \int_1^y |U_{\lambda+1}(t) - \ell|dt + \int_1^y |U_\lambda(t) - \ell|dt.$$

Consequently, by (3.3.7) and Theorem 3.3,

$$\int_1^y |u_\lambda(t)|dt = o(y), \quad \text{as } y \to \infty$$

Hence

$$(3.3.8) \qquad \int_o^y |s_\lambda(t) - \ell|dt = o(y), \text{ as } y \to \infty \;;$$

i.e.

$$s_n \to \ell \; [A_{\lambda-1}].$$

(ii) Suppose that $s_n \to \ell \; [A_{\lambda-1}]$ i.e. (3.3.8) holds. Again, by (2.2.5), it suffices to show that

$$\int_0^y |u_\lambda(t)|dt = o(y) \quad \text{as} \quad y \to \infty \;. \tag{3.3.9}$$

We have, by (2.2.1), that

$$\begin{aligned} u_\lambda(t) &= (1+t)^{-1}s_\lambda(t) - \lambda(1+t)^{-\lambda-1} \int_0^t (1+z)^{\lambda-1}s_\lambda(z)dz \\ &= (1+t)^{-1}[s_\lambda(t) - \ell] - \lambda(1+t)^{-\lambda-1} \int_0^t (1+z)^{\lambda-1}[s_\lambda(z) - \ell]dz \\ &\qquad + \ell\,(1+t)^{-\lambda-1}, \end{aligned}$$

and so

$$\begin{aligned} &\int_0^y |u_\lambda(t)|dt \\ &\quad \le \int_0^y (1+t)^{-1}|s_\lambda(t) - \ell|dt \\ &\qquad\qquad + \lambda\int_0^y (1+t)^{-\lambda-1}dt \int_0^t (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz \\ &\qquad\qquad + |\ell| \int_0^y (1+t)^{-\lambda-1}dt \\ &\quad \le \int_0^y |s_\lambda(t) - \ell|dt + \lambda\int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz \int_z^y (1+t)^{-\lambda-1}dt \\ &\qquad\qquad + |\ell| \int_0^y (1+t)^{-\lambda-1}dt. \end{aligned}$$

Hence, by (3.3.3) and (3.3.8)

$$\int_0^y |u_\lambda(t)|dt = o(y) + I(y),$$

where

$$I(y) = \lambda\int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz \int_z^y (1+t)^{-\lambda-1}dt$$

$$= \int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz\ [(1+z)^{-\lambda} - (1+y)^{-\lambda}]$$

$$= \int_0^y (1+z)^{-1}|s_\lambda(z) - \ell|dz - (1+y)^{-\lambda} \int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz$$

$$\leq \int_0^y |s_\lambda(z) - \ell|dz - (1+y)^{-\lambda} \int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz$$

$$= o(y) - I_1(y).$$

Since

$$I_1(y) = (1+y)^{-\lambda} \int_0^y (1+z)^{\lambda-1}|s_\lambda(z) - \ell|dz$$

$$= (1+y)^{-\lambda} \int_0^y (1+z)^{\lambda} (1+z)^{-1}|s_\lambda(z) - \ell|dz$$

$$\leq \int_0^y (1+z)^{-1}|s_\lambda(z) - \ell|dz$$

$$\leq \int_0^y |s_\lambda(z) - \ell|dz$$

$$= o(y),$$

(3.3.9) follows.

Hence $s_n \to \ell\ [A'_\lambda]$.

This completes the proof of Theorem 3.7.

CHAPTER 4

STRONG ABEL-TYPE SUMMABILITY - II

In this chapter, we consider the strong Abel-type summability methods $[A_\lambda]_p$ and $[A'_\lambda]_p$, which, for p = 1, reduce to the methods $[A_\lambda]$ and $[A'_\lambda]$ investigated in the preceeding chapter. It is shown that these methods have analogous properties.

4.1 *DEFINITIONS*

Let $s_\lambda(y)$, $u_\lambda(y)$ and $U_\lambda(y)$ be as defined in §1.2. Let p be a positive real number.

Strong Abel-type summability with index p. $[A_\lambda]_p$.

If

$$\int_0^y |s_{\lambda+1}(t) - \ell|^p \, dt = o(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A_λ*-convergent with index* p or $[A_\lambda]_p$*-convergent* to ℓ and write

$$s_n \to \ell \; [A_\lambda]_p \; .$$

Strong Abel-type summability with index p. $[A_\lambda']_p$.

If

$$\int_0^y |U_{\lambda+1}(t) - \ell|^p dt = o(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A_λ*-convergent with index* p or $[A_\lambda']_p$*-convergent* to ℓ and write

$$s_n \to \ell \ [A_\lambda']_p.$$

Strong boundedness with index p.

If

$$\int_0^y |s_{\lambda+1}(t)|^p dt = O(y)$$

as $y \to \infty$, we say that the sequence $\{s_n\}$ is *strongly* A_λ*-bounded with index* p or $[A_\lambda]_p$*-bounded* and write

$$s_n = O(1) \ [A_\lambda]_p .$$

Similarly we define $[A_\lambda']_p$-boundedness.

4.2

In this section some consequences of our definitions are noted.

THEOREM 4.1

Let $0 < q < p$ *and* $s_n \to \ell\ [A'_\lambda]_p$, *then*

$$s_n \to \ell\ [A'_\lambda]_q$$

PROOF.

We have, by assumption that

$$\int_o^y |U_{\lambda+1}(t) - \ell|^p dt = o(y), \text{ as } y \to \infty.$$

Using Hölder's inequality, with indices $\frac{p}{q}$ and $\frac{p}{p-q}$, we have

$$\int_o^y |U_{\lambda+1}(t) - \ell|^q dt \leq \left(\int_o^y |U_{\lambda+1}(t) - \ell|^p dt\right)^{q/p} \left(\int_o^y dt\right)^{1-q/p}$$

$$= o(y^{q/p})\ O(y^{1-q/p})$$

$$= o(y),$$

as $y \to \infty$.

The theorem follows.

COROLLARY 4.1.1

Let $0 < q < p$ *and* $s_n = O(1)\ [A'_\lambda]_p$, *then* $s_n = O(1)\ [A'_\lambda]_q$.

THEOREM 4.2

Let $\lambda > 0$, $p > 1$ *and* $s_n \to \ell\ [A'_\lambda]_p$. *Then*

$$s_n \to \ell\ (A'_\lambda)$$

PROOF.

By Theorem 4.1, we have that

$$s_n \to \ell\ [A'_\lambda]_p \Rightarrow s_n \to \ell\ [A'_\lambda],$$

and by Theorem 3.1, that

$$s_n \to \ell\ [A'_\lambda] \Rightarrow s_n \to \ell\ (A'_\lambda).$$

Hence the theorem follows.

COROLLARY 4.2.1

Let $\lambda > 0$, $p > 1$ *and* $s_n = O(1)\ [A'_\lambda]_p$. *Then*

$$s_n = O(1)\ (A'_\lambda).$$

THEOREM 4.3

If $s_n \to \ell\ (A'_\lambda)$, *then* $s_n \to \ell\ [A'_{\lambda-1}]_p$ *for every* $p > 0$.

PROOF.

We have, by assumption, that

$$|U_\lambda(t) - \ell|^p = o(1) \quad \text{as } t \to \infty\ ,$$

and so, by the regularity of (C, 1)-method, that

$$\int_o^y |U_\lambda(t) - \ell|^p dt = o(y) \quad \text{as } y \to \infty\ .$$

The theorem follows.

COROLLARY 4.3.1

If $s_n = O(1)$ (A'_λ), *then* $s_n = O(1)$ $[A'_{\lambda-1}]_p$ *for every* $p > 0$.

The following theorem is an immediate consequence of Theorem 4.2 and Theorem 4.3.

THEOREM 4.4

Let $\lambda > 0$, $p > 1$ *and* $s_n \to \ell$ $[A'_\lambda]_p$. *Then*

$$s_n \to \ell [A'_{\lambda-1}]_p.$$

COROLLARY 4.4.1

Let $\lambda > 0$, $p > 1$ *and* $s_n = O(1)$ $[A'_\lambda]_p$. *Then*

$$s_n = O(1)\ [A'_{\lambda-1}]_p\ .$$

The next theorem gives the necessary and sufficient conditions for the $[A'_\lambda]_p$-convergence.

THEOREM 4.5

For $\lambda > 0$ *and* $p > 1$, *the necessary and sufficient conditions for the* $[A'_\lambda]_p$ *-convergence of the sequence* $\{s_n\}$ *to* ℓ *are that*

$$(4.2.1) \qquad s_n \to \ell\ (A'_\lambda)$$

and

$$(4.2.2) \qquad \int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right|^p dt = o(y) \text{ as } y \to \infty$$

PROOF.

Necessity:

Suppose that

$$s_n \to \ell \ [A'_\lambda]_p$$

Then (4.2.1) follows from Theorem 4.2.

Now, by (2.2.6'), we have that

$$\left|t \frac{d}{dt} U_\lambda(t)\right|^p = \frac{1}{\lambda^p} \left|U_{\lambda+1}(t) - U_\lambda(t)\right|^p$$

$$\leq M\left\{\left|U_{\lambda+1}(t) - \ell\right|^p + \left|U_\lambda(t) - \ell\right|^p\right\} ,$$

and so

$$\int_0^y \left|t \frac{d}{dt} U_\lambda(t)\right|^p dt \leq M \int_0^y \left|U_{\lambda+1}(t) - \ell\right|^p dt + M \int_0^y \left|U_\lambda(t) - \ell\right|^p dt$$

$$= o(y), \quad \text{as } y \to \infty,$$

by Theorem 4.4.

This completes the proof of the necessity part.

Sufficiency:

Suppose that (4.2.1) and (4.2.2) hold.

Now, by (4.2.1) and Theorem 4.3, we have that

$$\int_0^y \left|U_\lambda(t) - \ell\right|^p dt = o(y), \quad \text{as } y \to \infty .$$

Hence, it follows from (4.2.2) and (2.2.6'), that

$$\int_0^y |U_{\lambda+1}(t) - \ell|^p dt$$

$$\leq M \int_0^y |t \frac{d}{dt} U_\lambda(t)|^p \, dt + M \int_0^y |U_\lambda(t) - \ell|^p \, dt$$

$$= o(y),$$

as $y \to \infty$.

The theorem follows.

COROLLARY 4.5.1

For $\lambda > 0$, $p > 1$, $s_n = O(1)$ $[A'_\lambda]$ *if and only if*

$$s_n = O(1) \ (A'_\lambda)$$

and

$$\int_0^y |t \frac{d}{dt} U_\lambda(t)|^p dt = O(y) \quad \text{as} \quad y \to \infty .$$

In a manner similar to our Theorem 4.5, we can establish the following

THEOREM 4.5'

For $\lambda > -1$ *and* $p > 1$, *the necessary and sufficient conditions for the* $[A_\lambda]_p$*-convergence of the sequence* $\{s_n\}$ *to* ℓ *are that*

(4.2.1') $$s_n \to \ell \ (A_\lambda)$$

and

(4.2.2') $$\int_0^y |t \frac{d}{dt} s_\lambda(t)|^p dt = o(y), \quad \text{as } y \to \infty.$$

4.3

In this section we prove the $[A_\lambda]_p$ and $[A'_\lambda]_p$-summability analogues of Theorem A and Theorem B, for the case $p > 1$.

THEOREM 4.6

Let $\lambda > 0$, $p > 1$. *Then* $s_n \to \ell\ [A_\lambda]_p$ *if and only if* $s_n \to \ell\ [A'_\lambda]_p$ *and* $nu_n \to 0\ [A_{\lambda-1}]_p$.

PROOF.

(i) Suppose that $s_n \to \ell\ [A_\lambda]_p$ i.e.

$$\int_0^y |s_{\lambda+1}(t) - \ell|^p dt = o(y), \text{ as } y \to \infty. \tag{4.3.1}$$

Now, by (2.2.3), we have that

$$U_{\lambda+1}(t) - \ell$$

$$= (\lambda+1)(1+t)^{-\lambda-1}\int_0^t (1+z)^\lambda[s_{\lambda+1}(z) - \ell]dz - \ell(1+t)^{-\lambda-1}$$

Hence

$$|U_{\lambda+1}(t) - \ell|^p$$

$$\leq 2^p\left\{(\lambda+1)^p(1+t)^{-p(\lambda+1)}\left|\int_0^t (1+z)^\lambda[s_{\lambda+1}(z) - \ell]dz\right|^p + |\ell|^p\ (1+t)^{-p(\lambda+1)}\right\},$$

and so

$$\int_0^y |U_{\lambda+1}(t) - \ell|^p dt \le M \int_0^y (1+t)^{-p(\lambda+1)} dt \left|\int_0^t (1+z)^\lambda [s_{\lambda+1}(z) - \ell] dz\right|^p$$

$$+ M \int_0^y (1+t)^{-p(\lambda+1)} dt.$$

$$= M \int_0^y J_1(t)dt + M \int_0^y J_2(t)dt, \quad \text{(say)}$$

Now

$$\int_0^y J_2(t)dt = \int_0^y (1+t)^{-p(\lambda+1)} dt = M[(1+y)^{-p(\lambda+1)+1} - 1]$$

Since $-p(\lambda+1)+1 < 0$, we have

$$(4.3.2) \quad \int_0^y J_2(t)dt = O(1) = o(y) \quad \text{as} \quad y \to \infty .$$

Further,

$$J_1(t) = (1+t)^{-p(\lambda+1)} \left|\int_0^t (1+z)^\lambda [s_{\lambda+1}(z) - \ell] dz\right|^p$$

Using Hölder's inequality with indexes p and p/(p-1), we have

$$J_1(t) \le (1+t)^{-p(\lambda+1)} \left|\int_0^t |s_{+1}(z) - \ell|^p dz\right| \left|\int_0^t (1+z)^{\frac{\lambda p}{p-1}} dz\right|^{p-1}$$

$$= M(1+t)^{-p(\lambda+1)} \left|\int_0^t s_{\lambda+1}(z) - \ell|^p dz\right| \left|(1+t)^{\frac{\lambda p+p-1}{p-1}} - 1\right|^{p-1}$$

$$\leq M\ (1+t)^{-p(\lambda+1)} \left| \int_o^t |s_{\lambda+1}(z) - \ell|^p dz \right| \left| (1+t)^{\lambda p+p-1} + 1 \right|$$

$$= M\ (1+t)^{-1} \int_o^t |s_{\lambda+1}(z) - \ell|^p dz + M\ (1+t)^{-p(\lambda+1)} \int_o^t |s_{\lambda+1}(z) - \ell|^p dz$$

$$= J_1'(t) + J_1''(t),\ (\text{say}).$$

Now

$$\int_o^y J_1'(t)\ dt = M \int_o^y (1+t)^{-1} dt \int_o^t |s_{\lambda+1}(z) - \ell|^p dz$$

$$= M \int_o^y \frac{t}{1+t}\ dt\ \frac{1}{t} \int_o^t |s_{\lambda+1}(z) - \ell|^p dz$$

$$= M \int_o^y f(t)\ \frac{t}{1+t}\ dt,$$

where

(4.3.3) $\qquad f(t) = \frac{1}{t} \int_o^t |s_{\lambda+1}(z) - \ell|^p dz = o(1)$ as $t \to \infty$, by

(4.3.1).

Hence

$$\frac{1}{y} \int_o^y J_1'(t)\ dt = M \int_o^\infty k(y,\ t)\ f(t)\ dt,$$

where

$$k(y,t) = \frac{1}{y}\ \frac{t}{1+t} \qquad t \leq y$$

(4.3.4)

$$= 0 \qquad t > y$$

It is easily verified that the kernel $k(y, t)$ defines a zero-preserving transformation. For, it is enough to verify that

$$(4.3.5) \qquad \int_0^\infty |k(y, t)|dt < M, \quad \text{where } M \text{ is independent of } y$$

and

$$(4.3.6) \qquad \int_0^T |k(y, t)|dt \to 0 \quad \text{when } y \to \infty, \text{ for every finite } T.$$

Now,

$$\int_0^\infty |k(y, t)|dt = \frac{1}{y}\int_0^y \frac{t}{1+t}\,dt$$

$$\leq \frac{1}{y}\int_0^y dt = 1,$$

and so (4.3.5) holds.

Further,

$$\int_0^T |k(y, t)|dt = \frac{1}{y}\int_0^T \frac{t}{1+t}\,dt$$

$$\leq \frac{T}{y} \to 0 \quad \text{as} \quad y \to \infty.$$

Hence (4.3.6) is satisfied.

It follows that

$$(4.3.7) \qquad \frac{1}{y}\int_0^y J_1'(t)dt = o(1), \quad \text{as } y \to \infty$$

Similarly, for $J_1''(t)$, we have that

$$\frac{1}{y}\int_0^y J_1''(t)dt = \frac{M}{y}\int_0^y \frac{t}{(1+t)^{p(\lambda+1)}}\, dt\, \frac{1}{t}\int_0^t |s_{\lambda+1}(z) - \ell|^p dz$$

$$= M\int_0^\infty k(y, t)\, f(t)\, dt,$$

where $f(t)$ is given by (4.3.3) and

$$(4.3.8) \qquad k(y, t) = \begin{cases} \dfrac{1}{y}\dfrac{t}{(1+t)^{p(\lambda+1)}} & t \leq y \\ = 0 & t > y \end{cases}$$

Again, it is easily verified that the kernel $k(y, t)$ satisfies the conditions (4.3.5) and (4.3.6) and hence it defines a zero preserving transformation.

It follows that

$$(4.3.9) \qquad \frac{1}{y}\int_0^y J_1''(t)dt = o(1), \text{ as } y \to \infty.$$

Collecting the results (4.3.2), (4.3.7) and (4.3.9), we have that

$$(4.3.10) \qquad \int_0^y |U_{\lambda+1}(t) - \ell|^p dt = o(y), \quad \text{as } y \to \infty;$$

i.e.

$$s_n \to \ell\ [A'_\lambda]_p.$$

To complete the proof of (i) it remains to show that $nu_n \to 0\ [A_{\lambda-1}]_p$ Since

$$z \frac{d}{dz} s_\lambda(z) = t_\lambda(z),$$

by (1.3.1), we have from (4.2.2') that (4.3.1) implies

$$nu_n \to 0\ [A_{\lambda-1}]_p \ . \tag{4.3.11}$$

This completes the proof of (i).

(ii) Suppose that (4.3.10) and (4.3.11) hold.

We first show that

$$s_n \to \ell\ [A'_\lambda]_p \Rightarrow s_n \to \ell\ [A_{\lambda-1}]_p \ . \tag{4.3.12}$$

We have, by (2.2.5), that

$$|s_\lambda(t) - \ell|^p \le M\left\{|U_{\lambda+1}(t) - \ell|^p + |u_\lambda(t)|^p\right\},$$

and so

$$\int_0^y |s_\lambda(t) - \ell|^p dt \le M \int_0^y |U_{\lambda+1}(t) - \ell|^p dt + M \int_0^y |u_\lambda(t)|^p dt.$$

Hence, to establish (4.3.12), it remains to show that

$$\int_0^y |u_\lambda(t)|^p\, dt = o(y), \quad \text{as } y \to \infty.$$

Now

$$\int_0^1 |u_\lambda(t)|^p dt = o(y), \quad \text{as } y \to \infty,$$

and

$$\int_1^y |u_\lambda(t)|^p dt = \int_1^y \frac{1}{t^p} |U_{\lambda+1}(t) - U_\lambda(t)|^p \, dt,$$

by (2.2.6).

Hence

$$\int_1^y |u_\lambda(t)|^p dt \leq M \int_1^y |U_{\lambda+1}(t) - \ell|^p dt + M \int_1^y |U_\lambda(t) - \ell|^p dt$$

$$= o(y),$$

by assumption in (4.3.12) and Theorem 4.4.

Hence (4.3.12) follows.

Further, it follows from (1.2.1), that

$$\int_0^y |s_{\lambda+1}(z) - \ell|^p \, dz \leq M\int_0^y |t_\lambda(z)|^p dz + M \int_0^y |s_\lambda(z) - \ell|^p dz.$$

It follows from (4.3.11) and (4.3.12) that

$$\int_0^y |s_{\lambda+1}(z) - \ell|^p dz = o(y), \quad \text{as } \quad y \to \infty$$

i.e.

$$s_n \to \ell \; [A_\lambda]_p.$$

Thus the proof of Theorem 4.6 is complete.

THEOREM 4.7

Let $\lambda > 0$, $p > 1$. *Then* $[A_\lambda']_p \simeq [A_{\lambda-1}]_p$.

PROOF.

(i) Suppose that $s_n \to \ell\ [A_\lambda']_p$.

We have already shown in (4.3.12) that

$$s_n \to \ell\ [A_{\lambda-1}]_p.$$

(ii) Suppose that $s_n \to \ell\ [A_{\lambda-1}]_p$ i.e.,

$$\int_0^y |s_\lambda(t) - \ell|^p dt = o(y). \tag{4.3.13}$$

We have, by (2.2.5), that

$$|U_{\lambda+1}(t) - \ell|^p \leq M\left\{|s_\lambda(t) - \ell|^p + |u_\lambda(t)|^p\right\},$$

and so

$$\int_0^y |U_{\lambda+1}(t) - \ell|^p\ dt \leq M \int_0^y |s_\lambda(t) - \ell|^p dt + M \int_0^y |u_\lambda(t)|^p dt.$$

Hence, to complete the proof of the theorem it remains to show that

$$\int_0^y |u_\lambda(t)|^p dt = o(y), \qquad \text{as} \quad y \to \infty$$

Now, by (2.2.1), we have that

$$u_\lambda(t) = (1+t)^{-1}[s_\lambda(t) - \ell] - \lambda(1+t)^{-\lambda-1}\int_o^t (1+z)^{\lambda-1}[s_\lambda(z) - \ell]dz$$
$$+ \ \ell(1+t)^{-\lambda-1},$$

and so

$$|u_\lambda(t)|^p$$
$$\leq M\Big\{(1+t)^{-p}|s_\lambda(t) - \ell|^p + \lambda^p(1+t)^{-p(\lambda+1)}\Big|\int_o^t (1+z)^{\lambda-1}[s_\lambda(z) - \ell]dz\Big|^p$$
$$+ \ |\ell|^p \ (1+t)^{-p(\lambda+1)} \ .$$

$$= M\ \Big\{I_1(t) + I_2(t) + I_3(t)\Big\},\ (\text{say}).$$

We consider the parts separately.

$$\int_o^y I_1(t)\ dt = \int_o^y (1+t)^{-p}\ |s_\lambda(t) - \ell|^p dt$$
$$\leq \int_o^y |s_\lambda(t) - \ell|^p\ dt,$$

and thus, by (4.3.13)

$$\int_o^y I_1(t)\ dt = o(y), \quad \text{as } y \to \infty \tag{4.3.14}$$

Now,

$$I_2(t) = (1+t)^{-p(\lambda+1)}\Big|\int_o^t (1+z)^{\lambda-1}[s_\lambda(z) - \ell]dz\Big|^p.$$

Using Hölder's inequality, with indexes p and $\frac{p}{p-1}$, we have

$$I_2(t) \leq (1+t)^{-p(\lambda+1)}\left|\int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz\right|\left|\int_o^t (1+z)^{-p/p-1} dz\right|^{p-1}$$

$$= M(1+t)^{-p(\lambda+1)}\left|\int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz\right|\left|(1+t)^{-1/p-1} - 1\right|^{p-1}$$

$$\leq M(1+t)^{-p(\lambda+1)}\left|\int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz\right|\left|(1+t)^{-1} + 1\right|$$

$$= M(1+t)^{-p(\lambda+1)-1}\int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz$$

$$+ M(1+t)^{-p(\lambda+1)}\int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz$$

$$\leq M[J_1(t) + J_2(t)], \quad \text{(say)}.$$

Now,

$$J_1(t) = (1+t)^{-p(\lambda+1)-1} \int_o^t (1+z)^{\lambda p}|s_\lambda(z) - \ell|^p dz$$

$$\leq (1+t)^{-p-1} \int_o^t |s_\lambda(z) - \ell|^p dz,$$

and thus

$$\int_o^y J_1(t)dt \leq \int_o^y (1+t)^{-p-1}dt \int_o^t |s_\lambda(z) - \ell|^p dz$$

$$= \int_o^y |s_\lambda(z) - \ell|^p dz \int_z^y (1+t)^{-p-1}dt$$

$$= M \int_o^y |s_\lambda(z) - \ell|^p dz \; [(1+z)^{-p} - (1+y)^{-p}]$$

$$= M \int_0^y (1+z)^{-p} |s_\lambda(z) - \ell|^p dz + M(1+y)^{-p} \int_0^y |s_\lambda(z) - \ell|^p dz$$

$$\leq M \int_0^y |s_\lambda(z) - \ell|^p dz + M \int_0^y |s_\lambda(z) - \ell|^p dz.$$

Hence, by (4.3.13), we get

(4.3.15) $$\int_0^y J_1(t)dt = o(y), \quad \text{as } y \to \infty,$$

Also, we have

$$J_2(t) = (1+t)^{-p(\lambda+1)} \int_0^t (1+z)^{\lambda p} |s_\lambda(z) - \ell|^p dz$$

$$\leq (1+t)^{-p} \int_0^t |s_\lambda(z) - \ell|^p dz,$$

and so

$$\int_0^y J_2(t)dt \leq \int_0^y (1+t)^{-p} dt \int_0^t |s_\lambda(z) - \ell|^p dz$$

$$= \int_0^y |s_\lambda(z) - \ell|^p dz \int_z^y (1+t)^{-p} dt$$

$$\leq M \int_0^y (1+z)^{-p+1} |s_\lambda(z) - \ell|^p dz$$

$$+ M(1+y)^{-p+1} \int_0^y |s_\lambda(z) - \ell|^p dz$$

$$\leq M \int_0^y |s_\lambda(z) - \ell|^p dz,$$

since $-p + 1 < 0$.

It follows that

$$(4.3.16) \qquad \int_o^y J_2(t)dt = o(y), \quad \text{as } y \to \infty$$

It remains to consider $I_3(t)$.

$$\int_o^y I_3(t)\ dt = \int_o^y (1+t)^{-p(\lambda+1)}dt$$

$$= M[(1+y)^{-p(\lambda+1)+1} - 1]$$

$$= O(1), \text{ as } y \to \infty, \text{ since } -p(\lambda+1)+1 < 0.$$

Hence

$$(4.3.17) \qquad \int_o^y I_3(t)dt = o(y), \quad \text{as } y \to \infty .$$

It follows, from (4.3.14), (4.3.15), (4.3.16) and (4.3.17) that

$$\int_o^y |u_\lambda(t)|^p dt = o(y) \qquad \text{as} \quad y \to \infty .$$

The theorem follows.

CHAPTER 5

THE PRODUCT METHOD $A_\lambda H_\chi$

In this chapter the method $A_\lambda H_\chi$—the iteration product of the Abel-type method A_λ and the Hausdorff method H_χ is considered.

Though the study of the product methods could be traced back to the works of Hausdorff [17]; Lord [20] and Agnew [1], the real beginnings seem to have been made with the publication of two papers by O. Szasz [31] and [32]. Subsequently, the product of different methods has been investigated by various authors, including A. Amir (Jakimovski) [2]; C.T. Rajagopal [26]; M.S. Ramanujan [27] and [28]; D. Borwein [4], [7] and [8] and Kazuo Ishiguro [18] and [19].

The product method $A_\lambda H_\chi$ has been dealt, in special cases by Lord [20] ($\lambda = 0$ and $H_\chi = (C, \alpha)$) and Szasz [32] ($\lambda = 0$) and in general case by Amir (Jakimovski) [3] and Borwein [4], [7] and [8]. The absolute and the strong summability methods based upon the product method $A_\lambda H_\chi$ are defined and some of their properties are investigated in this chapter.

5.1 *DEFINITIONS*

Let $\{\mu_n\}$ be a sequence of real numbers and let

$$(5.1.1) \qquad h_n = \sum_{r=o}^{n} \varepsilon_r^{n-r} s_r \sum_{m=o}^{n-r} (-1)^m \, \varepsilon_m^{n-r-m} \, \mu_{r+m}$$

Let the matrix of the linear transformation from s_n to h_n, as well as the transformation itself be denoted by H. We will write $h_n = H(s_n)$.

The method H

If $h_n \to \ell$, as $n \to \infty$, we say that the sequence $\{s_n\}$ is H-*convergent* to ℓ and write

$$s_n \to \ell \ (H).$$

Let $\chi(t)$ be a real-valued function of bounded variation in $[0, 1]$. If the sequence $\{\mu_n\}$ is the sequence of moments of the function $\chi(t)$, i.e. if

$$\mu_n = \int_o^1 t^n d\chi(t),$$

then, it is easily verified that (5.1.1) takes the form

$$(5.1.2) \qquad h_n = \sum_{r=o}^{n} \varepsilon_r^{n-r} \, s_r \int_o^1 t^r \, (1-t)^{n-r} \, d\chi(t).$$

In this case, we will denote the Hausdorff method H by H_χ. The following theorem for the regularity of the method H_χ is well-known. (See Hardy [14] §11.8)

THEOREM 5.1

The Hausdorff method H_χ *is regular if and only if*

(5.1.3) $$\chi(0+) = \chi(0)$$

and

(5.1.4) $$\chi(1) - \chi(0) = 1.$$

The product method $A_\lambda H_\chi$ is now defined in the obvious manner as under.

The method $A_\lambda H_\chi$

If $h_n \to \ell\ (A_\lambda)$, we say that the sequence $\{s_n\}$ is $A_\lambda H_\chi$-*convergent* to ℓ and write

$$s_n \to \ell\ (A_\lambda H_\chi).$$

The method $|A_\lambda H_\chi|$

If $h_n \to \ell\ |A_\lambda|$, we say that the sequence $\{s_n\}$ is $|A_\lambda H_\chi|$-*convergent* to ℓ and write

$$s_n \to \ell |A_\lambda H_\chi|.$$

The method $[A_\lambda H_\chi]_p$

If $h_n \to \ell\ [A_\lambda]_p$, we say that the sequence $\{s_n\}$ is $[A_\lambda H_\chi]_p$-*convergent* to ℓ and write

$$s_n \to \ell\ [A_\lambda H_\chi]_p\ .$$

5.2

The case $\lambda = 0$ of the following lemma is due to Szasz [31] and the case $\lambda > -1$ is due to Amir (Jakimovski) [3]. The proof given here is due to Borwein [4].

LEMMA 5.1

If $\lambda > -1$ *and* $\sum_{n=o}^{\infty} \varepsilon_n^{\lambda} s_n x^n$ *is convergent for* $0 \leq x < 1$, *then, for* $y > 0$

$$(5.2.1) \qquad (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n = \int_o^1 s_\lambda(yt)\,d\chi(t),$$

where h_n *is defined by (5.1.2).*

PROOF.

By (5.1.2), we have that

$$(1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n = \sum_{n=o}^{\infty} \frac{y^n}{(1+y)^{\lambda+1+n}} \varepsilon_n^{\lambda} \sum_{r=o}^{n} \varepsilon_r^{n-r} s_r \int_o^1 t^r (1-t)^{n-r} d\chi(t)$$

$$= \sum_{n=o}^{\infty} \frac{y^n}{(1+y)^{\lambda+1+n}} \varepsilon_n^{\lambda} \int_o^1 d\chi(t) \sum_{r=o}^{n} \varepsilon_r^{n-r} s_r t^r (1-t)^{n-r} .$$

Hence, assuming that the interchange is legitimate, we have that

$$(5.2.2) \quad (1+y)^{-\lambda-1} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n$$

$$= \int_o^1 d\chi(t) \sum_{n=o}^{\infty} \frac{y^n}{(1+y)^{\lambda+1+n}} \varepsilon_n^{\lambda} \sum_{r=o}^{n} \varepsilon_r^{n-r} s_r t^r (1-t)^{n-r} .$$

Now, assuming that the double sum is absolutely convergent, we can interchange the order of summation to obtain,

$$(5.2.3) \qquad (1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n$$

$$= \int_o^1 d\chi(t) \sum_{r=o}^{\infty} s_r t^r \sum_{n=r}^{\infty} \frac{y^n}{(1+y)^{\lambda+1+n}} \varepsilon_n^{\lambda} \varepsilon_r^{n-r} (1-t)^{n-r} .$$

Since

$$\varepsilon_n^{\lambda} \cdot \varepsilon_r^{n-r} = \varepsilon_r^{\lambda} \cdot \varepsilon_{n-r}^{\lambda+r} ,$$

it follows that

$$(1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n$$

$$= \int_o^1 d\chi(t) \sum_{r=o}^{\infty} \varepsilon_r^{\lambda} s_r \frac{(yt)^r}{(1+y)^{\lambda+1+r}} \sum_{n=r}^{\infty} \varepsilon_{n-r}^{\lambda+r} \left(\frac{y-yt}{1+y}\right)^{n-r} .$$

Now

$$\sum_{n=r}^{\infty} \varepsilon_{n-r}^{\lambda+r} \left(\frac{y-yt}{1+y}\right)^{n-r} = \sum_{n=o}^{\infty} \varepsilon_n^{\lambda+r} \left(\frac{y-yt}{1+y}\right)^n$$

$$= \left(1 - \frac{y-yt}{1+y}\right)^{-\lambda-r-1} ,$$

Hence

$$(5.2.4) \qquad \sum_{n=r}^{\infty} \varepsilon_{n-r}^{\lambda+r} \left(\frac{y-yt}{1+y}\right)^{n-r} = \left(\frac{1+yt}{1+y}\right)^{-\lambda-r-1} ,$$

and so

$$(1+y)^{-\lambda-1} \sum_{n=o}^{\infty} \varepsilon_n^{\lambda} h_n \left(\frac{y}{1+y}\right)^n = \int_o^1 d\chi(t) \sum_{r=o}^{\infty} \varepsilon_r^{\lambda} s_r \frac{(yt)^r}{(1+yt)^{\lambda+1+r}}$$

$$= \int_o^1 s_{\lambda}(yt)\, d\chi(t),$$

which is (5.2.1).

It remains to show that the inversions at (5.2.2) and (5.2.3) are justified.

We have, by (5.2.4), that

$$\sum_{r=o}^{\infty} \varepsilon_r^{\lambda} |s_r| \frac{(yt)^r}{(1+y)^{\lambda+1+r}} \sum_{n=r}^{\infty} \varepsilon_{n-r}^{\lambda+r} \left(\frac{y-yt}{1+y}\right)^{n-r}$$

$$= \sum_{r=o}^{\infty} \varepsilon_n^{\lambda} |s_r| \frac{(yt)^r}{(1+yt)^{\lambda+1+r}}$$

$$\leq \sum_{r=o}^{\infty} \varepsilon_n^{\lambda} |s_r| \left(\frac{yt}{1+yt}\right)^r .$$

Hence, for $0 \leq t \leq 1$, $y > 0$, we have

$$(5.3.5) \qquad \sum_{r=o}^{\infty} \varepsilon_r^{\lambda} |s_r| \frac{(yt)^r}{(1+yt)^{\lambda+1+r}} \leq \sum_{r=o}^{\infty} \varepsilon_r^{\lambda} |s_r| \left(\frac{y}{1+y}\right)^r < \infty ,$$

and so the inversion at (5.2.3) is legitimate.

The justification for inversion at (5.2.2) is an immediate consequence of (5.3.5) and $\int_o^1 |d\chi(t)| < \infty$.

This completes the proof of the lemma.

The following known result (see Amir (Jakimovski) [3] and Szasz [32]) is immediate from Lemma 5.1 and Hardy [14], Theorem 217.

THEOREM 5.2

If $\lambda > -1$, H_χ *is a regular Hausdorff method and* $s_n \to \ell\ (A_\lambda)$, *then* $s_n \to \ell\ (A_\lambda H_\chi)$.

5.3

In this section we consider the absolute and strong summability analogues of Theorem 5.2.

THEOREM 5.3

If $\lambda > -1$, H_χ *is a regular Hausdorff method and* $s_n \to \ell\ |A_\lambda|$, *then* $s_n \to \ell\ |A_\lambda H_\chi|$.

PROOF.

Let

$$(5.3.1) \qquad h_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda\, h_n \left(\frac{y}{1+y}\right)^n ,$$

where h_n is defined by (5.1.2) and let V be the total variation of $s_\lambda(y)$ in $[0, \infty)$.

We have for $0 \le y_0 < y_1 < \dots < y_n$, by (5.2.1), that

$$\sum_{r=1}^{\infty} |h_\lambda(y_r) - h_\lambda(y_{r-1})| = \sum_{r=1}^{n} \left| \int_0^1 \left\{ s_\lambda(y_r t) - s_\lambda(y_{r-1} t) \right\} d\chi(t) \right|$$

$$\le \sum_{r=1}^{n} \int_0^1 |s_\lambda(y_r t) - s_\lambda(y_{r-1} t)|\, |d\chi(t)|$$

$$= \int_0^1 \sum_{r=1}^{n} |s_\lambda(y_r t) - s_\lambda(y_{r-1} t)||d\chi(t)|$$

$$\leq V \int_0^1 |d\chi(t)|$$

$$< \infty \qquad ,$$

It follows that $s_n \to \ell' \; |A_\lambda H_\chi|$ (say). That $\ell' = \ell$ is a consequence of Theorem 5.2.

The argument above also yields the following theorem (Cf. Theorem 217. Hardy [14]).

THEOREM 5.4

Let $\chi(t)$ *be of bounded variation in* $[0, 1]$ *and* $f(x)$ *be of bounded variation in* $[0, \infty)$. *Let*

$$g(x) = \int_0^1 f(xt) \, d\chi(t)$$

be defined for all $x \geq 0$. *Then* $g(x)$ *is of bounded variation in* $[0, \infty)$.

THEOREM 5.5

If $\lambda > -1$, H_χ *is a regular Hausdorff method and* $s_n \to \ell \; [A_\lambda]$, *then* $s_n \to \ell \; [A_\lambda H_\chi]$.

PROOF.

Let $h_{\lambda+1}(y)$ be as defined in (5.3.1) with $\lambda+1$ in place of λ. Then, we have, by (5.2.1), that

$$h_{\lambda+1}(y) - \ell = \int_0^1 s_{\lambda+1}(yt)\, d\chi(t) - \ell$$

$$= \int_0^1 \left\{ s_{\lambda+1}(yt) - \ell \right\} d\chi(t),$$

since, by (5.1.5)

$$\int_0^1 d\chi(t) = \chi(1) - \chi(0) = 1.$$

Thus

$$\int_0^y |h_{\lambda+1}(z) - \ell|\, dz = \int_0^y dz \left| \int_0^1 \left\{ s_{\lambda+1}(zt) - \ell \right\} d\chi(t) \right|$$

$$\leq \int_0^y dz \int_0^1 |s_{\lambda+1}(zt) - \ell|\ |d\chi(t)|$$

$$= \int_0^1 |d\chi(t)| \int_0^y |s_{\lambda+1}(zt) - \ell|\, dz$$

$$= y \int_0^1 |d\chi(t)|\ \frac{1}{yt} \int_0^{yt} |s_{\lambda+1}(x) - \ell|\, dx\ .$$

Hence

$$\frac{1}{y} \int_0^y |h_{\lambda+1}(z) - \ell|\, dz \leq \int_0^1 f(yt)\ |d\chi(t)|,$$

where

$$f(t) = \frac{1}{t} \int_0^t |s_{\lambda+1}(x) - \ell|\, dx = o(1) \text{ as } t \to \infty$$

Also $f(t) = O(1)$ as $t \to 0+$.

Choose T such that $|f(t)| < \varepsilon$ for $t \geq T$, and let

$$M(T) = \sup_{t \leq T} |f(t)| .$$

It follows that for $y > T$, we have

$$\frac{1}{y} \int_o^y |h_{\lambda+1}(z) - \ell| dz \leq \int_o^{T/y} |f(yt)||d\chi(t)| + \int_{T/y}^1 |f(yt)||d\chi(t)|$$

$$\leq M(T) \int_o^{T/y} |d\chi(t)| + \varepsilon \int_o^1 |d\chi(t)|$$

$$= o(1), \quad \text{as } y \to \infty ,$$

since $\chi(t)$ is continuous at 0, by (5.1.3).

The theorem follows.

THEOREM 5.6

If $\lambda > -1$, $p > 1$, H_χ *is a regular Hausdorff method and* $s_n \to \ell\ [A_\lambda]_p$, *then* $s_n \to \ell\ [A_\lambda H_\chi]_p$.

PROOF.

We have, as in the preceeding theorem,

$$|h_{\lambda+1}(z) - \ell|^p = \left| \int_o^1 \left\{ s_{\lambda+1}(zt) - \ell \right\} d\chi(t) \right|^p$$

$$\leq \left\{ \int_o^1 |s_{\lambda+1}(zt) - \ell| \ |d\chi(t)| \right\}^p$$

Now, by Hölder's inequality for Stieltjes integrals (See Hardy, [15] Theorem 210) we have that

$$|h_{\lambda+1}(z) - \ell|^p \leq \left(\int_0^1 |s_{\lambda+1}(zt) - \ell|^p |d\chi(t)|\right)\left(\int_0^1 |d\chi(t)|\right)^{p-1}$$

$$= M \int_0^1 |s_{\lambda+1}(zt) - \ell|^p |d\chi(t)|.$$

Hence

$$\int_0^y |h_{\lambda+1}(z) - \ell|^p dz \leq M \int_0^y dz \int_0^1 |s_{\lambda+1}(zt) - \ell|^p \, |d\chi(t)|,$$

and the proof follows as in Theorem 5.5.

This completes the proof of the theorem.

CHAPTER 6

THE LOGARITHMIC METHOD OF SUMMABILITY L

In this chapter the logarithmic method of summability L is studied. This method is considered by Borwein [7] and [8], and has also been applied to the summability of Fourier series by various authors, e.g., Mohanty and Patnaik [24]. The main results proved in this chapter provide the absolute summability analogues of Theorems 1, 2 and 5 of Borwein [8].

6.1 *DEFINITIONS*

The logarithmic method L.

If

$$(6.1.1) \qquad L(x) = -\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

tends to a finite limit ℓ as $x \to 1$ in the open interval $(0, 1)$, we say that the sequence $\{s_n\}$ is L-*convergent* to ℓ and write

$$s_n \to \ell \ (L).$$

REMARKS.

1. The L-method is a special case of the class of summability methods J as given in Hardy [14] §4.12. (Take $p_n = \frac{1}{n+1}$). It follows that this method is regular.

2. Borwein [7] has constructed a scale of logarithmic summability methods (L, α) and the method L is the case $\alpha = 1$.

Let $\lambda > -1$ and $\{s_n^\lambda\}$ be the sequence of Cesàro means of order λ of the sequence $\{s_n\}$; i.e.,

$$s_n^\lambda = \frac{1}{\varepsilon_n^\lambda} \sum_{r=o}^{n} \varepsilon_{n-r}^{\lambda-1} s_r \ .$$

The method (A, λ).

If

$$s^\lambda(x) = (1-x) \sum_{n=o}^{\infty} s_n^\lambda x^n \tag{6.1.2}$$

tends to a finite limit ℓ as $x \to 1$ in the open interval $(0, 1)$, we say that the sequence $\{s_n\}$ is (A, λ)-*convergent* to ℓ and write

$$s_n \to \ell \ (A, \lambda).$$

REMARKS.

1. The method $(A, 0)$ is the ordinary Abel method A. We will also write $s(x)$ in place of $s^o(x)$.

2. The method (A, λ), which is evidently the product method $A(C, \lambda)$, was first introduced by Lord in [20], but was subsequently redefined by Amir (Jakimovski) in [2]. It has also been considered by Borwein in [7] and [8].

The absolute logarithmic summability $|L|$.

If $L(x)$ is of bounded variation in $(0, 1)$ and tends to the limit ℓ as $x \to 1$ in $(0, 1)$, we say that the sequence $\{s_n\}$ is *absolutely* L-*convergent* or $|L|$-*convergent* to ℓ and write

$$s_n \to \ell \ |L|.$$

REMARK.

It is easily verified that $L(x)$ is of bounded variation in $(0, \delta]$ for $\delta < 1$. Hence, it would be enough to consider the bounded variability of $L(x)$ in $[\delta, 1)$.

The absolute summability $|A, \lambda|$.

If $s^{\lambda}(x)$ is of bounded variation in $(0, 1)$ and tends to ℓ as $x \to 1$ in $(0, 1)$, we say that the sequence $\{s_n\}$ is *absolutely* (A, λ)-*convergent* or $|A, \lambda|$-*convergent* to ℓ and write

$$s_n \to \ell \ |A, \lambda|.$$

We define the product method LH_χ, where H_χ is a regular Hausdorff method as under:

Let $\{h_n\}$ be the sequence of Hausdorff transforms of the sequence $\{s_n\}$, i.e.

(6.1.3) $$h_n = \sum_{r=0}^{n} \varepsilon_r^{n-r} s_r \int_0^1 t^r(1-t)^{n-r} d\chi(t)$$

The product method LH_χ.

If $h_n \to \ell$ (L), we say that the sequence $\{s_n\}$ is LH_χ-*convergent* to ℓ and write

$$s_n \to \ell \; (LH_\chi).$$

The method $|LH_\chi|$.

If $h_n \to \ell \; |L|$, we say that the sequence $\{s_n\}$ is $|LH_\chi|$-*convergent* to ℓ and write

$$s_n \to \ell \; |LH_\chi|.$$

6.2

The following theorem is known (See Hardy [14] page 81). Hardy, however, does not give a proof, so a proof is given here. See also Borwein [5] for a more general result.

THEOREM 6.1

If $s_n \to \ell$ (A), *then* $s_n \to \ell$ (L).

PROOF.

Let $L(x)$ and $s(x)$ be as defined by (6.1.1) and (6.1.2) respectively. It follows that

(6.2.1) $$L(x) = -\frac{1}{\log(1-x)} \int_0^x \frac{s(t)}{1-t} dt$$

Since $s_n \to \ell$ (A), we have that, for $\varepsilon > 0$, $\exists y < 1$, such that

$$(6.2.2) \qquad |s(x) - \ell| < \varepsilon \qquad \text{for } 1 > x > y.$$

Hence, by (6.2.1), we have that

$$L(x) - \ell = -\frac{1}{\log(1-x)} \int_0^x \frac{s(t)}{1-t}\, dt - \ell$$

$$= -\frac{1}{\log(1-x)} \int_0^x \frac{s(t)-\ell}{1-t}\, dt ,$$

and so for $x > y$, we have, by (6.2.2),

$$|L(x) - \ell| \leq \frac{1}{|\log(1-x)|} \int_0^y \frac{|s(t)-\ell|}{1-t}\, dt + \frac{\varepsilon}{|\log(1-x)|} \int_y^x \frac{dt}{1-t}$$

$$\leq \frac{1}{|\log(1-x)|} \int_0^y \frac{|s(t)-\ell|}{1-t}\, dt + \frac{\varepsilon}{|\log(1-x)|} \int_0^x \frac{dt}{1-t}$$

$$\leq \frac{1}{|\log(1-x)|} \int_0^y \frac{|s(t)-\ell|}{1-t}\, dt + \varepsilon .$$

Now, since y is fixed, we have

$$\overline{\lim_{x\to 1}} \frac{1}{|\log(1-x)|} \int_0^y \frac{|s(t)-\ell|}{1-t}\, dt = 0,$$

and so

$$\overline{\lim_{x\to 1}} |L(x) - \ell| = 0,$$

i.e. $s_n \to \ell$ (L).

This completes the proof of Theorem 6.1.

COROLLARY 6.1.1

If $s_n = O(1)$ (A), *then* $s_n = O(1)$ (L).

By this we mean that $L(x) = O(1)$ for $0 < x < 1$, whenever $s(x) = O(1)$ for $0 < x < 1$.

LEMMA 6.1

If a *is real,*

(6.2.3) $$s_n \to \ell \ (L)$$

and

(6.2.4) $$(n+a)v_n = s_n \qquad (n = 0, 1, 2, \ldots),$$

then

(6.2.5) $$v_n \to 0 \ |L|.$$

PROOF.

Let

(6.2.6) $$\phi(x) = \sum_{n=m}^{\infty} \frac{s_n}{n+1} x^{n+a-1} \qquad (|x| < 1),$$

where $m > |a| + 2$.

We have, by (6.2.3), for $0 \leq x < 1$,

(6.2.7) $$|\phi(x)| < M|\log(1-x)|.$$

Let

$$\psi(x) = -\frac{1}{\log(1-x)} \sum_{n=m}^{\infty} \frac{v_n}{n+1} x^{n+1} . \tag{6.2.8}$$

Reasoning as in the proof of Lemma 2.2 (page 16) and taking note of Borwein's Lemma 1 in [7], it suffices to show that $\psi(x)$ is of bounded variation in $[\frac{1}{2}, 1)$.

Now, in virtue of (6.2.4) and (6.2.8), we have that

$$x^{1-a}\, \psi(x) = -\frac{1}{\log(1-x)} \int_0^x \phi(t)dt. \tag{6.2.9}$$

Evidently, it is enough to show that $x^{1-a}\,\psi(x)$ is of bounded variation in $[\frac{1}{2}, 1)$.

It follows from (6.2.9) that

$$\int_{1/2}^{1} \left|\frac{d}{dx}\left\{x^{1-a}\psi(x)\right\}\right| dx \le \int_{1/2}^{1} - d\left|\frac{1}{\log(1-x)}\right| \left|\int_0^x |\phi(t)|dt\right|$$

$$+ \int_{1/2}^{1} \left|\frac{1}{\log(1-x)}\right| |\phi(x)| dx$$

$$\le \int_0^1 - d\left|\frac{1}{\log(1-x)}\right| \left|\int_0^x |\phi(t)|dt\right| + \int_0^1 \left|\frac{1}{\log(1-x)}\right| |\phi(x)| dx.$$

Using (6.2.7), we get

$$\int_{1/2}^{1} \left|\frac{d}{dx}\left\{x^{1-a}\, \psi(x)\right\}\right| dx \le M \int_0^1 - d\left|\frac{1}{\log(1-x)}\right| \int_0^x |\log(1-t)| dt + M$$

$$= M \int_0^1 |\log(1-t)|\,dt \int_t^1 - d\left|\frac{1}{\log(1-x)}\right| + M$$

$$= M < \infty .$$

The lemma follows.

REMARK.

In the proof of the lemma we have actually used the weaker hypothesis $s_n = O(1)$ (L) than (6.2.3).

An immediate consequence is the following.

LEMMA 6.2

If p *and* q *are real and* $s_n \to \ell\ |L|$, *then*

$$\frac{n+p}{n+q}\ s_n \to \ell\ |L|.$$

PROOF.

By Lemma 6.1, we have

$$\frac{n+p}{n+q}\, s_n = s_n + \frac{p-q}{n+q}\, s_n \to \ell\ |L|.$$

We will require the following lemma, which is due to Borwein [8] (Lemma 3). For the sake of completeness the outlines of the proof are given here. (Cf. Lemma 5.1).

LEMMA 6.3

If

$$s(t) = \sum_{n=1}^{\infty} \frac{s_n}{n} \left(\frac{t}{1+t}\right)^n \tag{6.2.10}$$

and the series is convergent for all $t \geq 0$, *then, for* $y \geq 0$,

(6.2.11)
$$\sum_{n=1}^{\infty} \frac{h_n}{n} \left(\frac{y}{1+y}\right)^n$$
$$= \int_o^1 \left\{ s(yt) - s_o \log(1+yt) \right\} d\chi(t) + s_o \log(1+y) \int_o^1 d\chi(t),$$

where h_n *is given by (6.1.3).*

PROOF.

By (6.1.3), we have, for $y \geq 0$

$$\sum_{n=1}^{\infty} \frac{h_n}{n} \left(\frac{y}{1+y}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{y}{1+y}\right)^n \sum_{r=1}^{n} \varepsilon_r^{n-r} \, s_r \int_o^1 t^r (1-t)^{n-r} d\chi(t)$$
$$+ s_o \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{y}{1+y}\right)^n \int_o^1 (1-t)^n d\chi(t).$$

Now, the justifications for the inversions are similar to those given in the proof of Lemma 5.1 and hence are omitted here. By performing the inversions, we get

$$\sum_{n=1}^{\infty} \frac{h_n}{n} \left(\frac{y}{1+y}\right)^n = \int_o^1 d\chi(t) \sum_{r=1}^{\infty} \frac{s_r}{r} \left(\frac{yt}{1+y}\right)^r \sum_{n=r}^{\infty} \varepsilon_{r-1}^{n-1-r+1} \left(\frac{y-yt}{1+y}\right)^{n-r}$$

$$+ s_o \int_o^1 d\chi(t) \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{y-yt}{1+y}\right)^n$$

$$= \int_o^1 d\chi(t) \sum_{r=1}^{\infty} \frac{s_r}{r} \left(\frac{yt}{1+yt}\right)^r - s_o \int_o^1 \log \frac{1+yt}{1+y} \, d\chi(t)$$

$$= \int_o^1 \left\{ s(yt) - s_o \log(1+yt) \right\} d\chi(t)$$
$$+ s_o \log(1+y) \int_o^1 d\chi(t).$$

This proves the Lemma.

LEMMA 6.4

Let

$$\phi_t(y) = \frac{\log(1+yt)}{\log(1+y)} \quad . \tag{6.2.12}$$

Then $\phi_t(y)$ is of uniformly bounded variation with respect to y in the range $(0, \infty)$ for $0 \leq t \leq 1$.

PROOF.

We note that $\phi_t(y)$ is uniformly bounded in y in the range $(0, \infty)$ for $0 \leq t \leq 1$.

We show that $\phi_t(y)$ is monotonic increasing and hence of bounded variation.

By (6.2.12) we have that

$$\frac{d}{dy}\,\phi_t(y) = \frac{\frac{t}{1+yt}\,\log(1+y) - \frac{1}{1+y}\,\log(1+yt)}{\log^2(1+y)}$$

and so

$$\frac{d}{dy}\,\phi_t(y) = \frac{t(1+y)\log(1+y)-(1+yt)\log(1+yt)}{(1+y)(1+yt)\log^2(1+y)} \tag{6.2.13}$$

Now, for $y > 0$, $t > 0$ we have

$$\frac{(1+y)\log(1+y)}{y} - \frac{(1+yt)\log(1+yt)}{yt} = \psi(y) - \psi(yt), \text{ where} \tag{6.2.14}$$

$$\psi(y) = \frac{(1+y)\log(1+y)}{y}$$

$$= (1 + \frac{1}{y})\log(1+y),$$

and so

$$\psi'(y) = \log(1+y)(-\frac{1}{y^2}) + (1 + \frac{1}{y})\cdot\frac{1}{1+y}$$

$$= -\frac{\log(1+y)}{y^2} + \frac{1}{y} > 0,$$

for $y > 0$ if

$$-\log(1+y) + y > 0$$

i.e. if $y > \log(1+y)$

i.e. if $e^y > 1+y$,

which is true for $y > 0$.

It follows that $\psi(y)$ is monotonic increasing for $y > 0$.

Hence it follows from (6.2.14) that

$$\psi(y) - \psi(yt) > 0.$$

Hence we have, from (6.2.13), that $\phi_t(y)$ is an increasing function of y.

The lemma follows.

6.3

The following theorem, which provides the absolute summability analogue for Theorem 6.1, was obtained independently of Mohanty and Patnaik [24], who were led to a similar result in their study of absolute L-summability of Fourier series.

THEOREM 6.2

If $s_n \to \ell\ |A|$, *then* $s_n \to \ell\ |L|$.

PROOF.

We have, by (6.2.1), that

$$L(x) = -\frac{1}{\log(1-x)}\int_0^x \frac{s(t)}{1-t}\,dt \ .$$

Integrating by parts, we get that

$$L(x) = -\frac{1}{\log(1-x)}\left[-\log(1-t)s(t)\Big|_0^x + \int_0^x \log(1-t)s'(t)dt\right]$$

$$= s(x) - \frac{1}{\log(1-x)}\int_0^x \log(1-t)s'(t)dt .$$

Now

$$\int_{1/2}^1 \left| \frac{d}{dx}\left\{-\frac{1}{\log(1-x)}\int_0^x \log(1-t)s'(t)dt\right\}\right| dx$$

$$\leq \int_0^1 -d\left|\frac{1}{\log(1-x)}\right| \int_0^x |\log(1-t)||s'(t)|dt$$

$$+ \int_0^1 |s'(x)|dx$$

$$= \int_0^1 |\log(1-t)||s'(t)|dt \int_t^1 -d\left|\frac{1}{\log(1-x)}\right| + \int_0^1 |s'(x)|dx$$

$$= 2\int_0^1 |s'(t)|dt$$

$$< \infty \quad .$$

The theorem follows now by Theorem 6.1.

THEOREM 6.3

The method $|L|$ *is translative.*

PROOF.

Suppose that

$$(6.3.1) \qquad s_n \to \ell \ |L|$$

Now, we have, for $0 < x < 1$,

$$(6.3.2) \qquad -\frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1}$$

$$= -\frac{1}{x\log(1-x)} \sum_{n=1}^{\infty} \frac{s_n}{n+1} x^{n+1} - \frac{1}{x\log(1-x)} \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)} x^{n+1}$$

It follows, by (6.3.1) and Lemma 6.1 that the right hand side is of bounded variation in $[\frac{1}{2}, 1)$. Taking note of the translativity of L method (see Borwein [8] Lemma 1), we have that

$$s_{n+1} \to \ell \ |L|.$$

Similarly, we have, for $0 < x < 1$,

$$(6.3.3) \qquad -\frac{1}{x\log(1-x)} \sum_{n=1}^{\infty} \frac{s_{n-1}}{n+1} x^{n+1}$$

$$= -\frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_n}{n+1} x^{n+1} + \frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_n}{(n+2)(n+1)} x^{n+1} .$$

Again, applying Lemma 6.1, we get that the right hand side is of bounded variation in $[\frac{1}{2}, 1)$. Consequently

$$s_{n-1} \to \ell \ |L|.$$

This completes the proof of the lemma.

THEOREM 6.4

If $-1 < \lambda \leq 1$ *and* $s_n \to \ell \ |A,\lambda|$, *then*

$$s_n \to \ell \ |L|.$$

PROOF.

We have, by a known result (see Corollary 7.5.1) that

$$s_n \to \ell \ |A, \lambda| \Rightarrow s_n \to \ell \ |A, \lambda+\delta| \quad \text{for } \lambda > -1, \quad \delta > 0.$$

Thus, we may suppose that

$$(6.3.4) \qquad s_n \to \ell \ |A, 1|$$

Let $t_n = s_n^1$. Then by (6.3.4) we have that

$$(6.3.5) \qquad t_n \to \ell \ |A|$$

Now, by Theorem 6.2 , it follows that

$$t_n \to \ell \ |L|,$$

and so, by Theorem 6.3,

$$t_{n+1} \to \ell \ |L|. \tag{6.3.6}$$

It is easily verified that

$$s_{n+1} = t_{n+1} + (n+1)(t_{n+1} - t_n),$$

and therefore, we have, for $0 < x < 1$,

$$\begin{aligned}\frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1} &= \frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{t_{n+1}}{n+1} x^{n+1} + \frac{1-x}{\log(1-x)} \sum_{n=o}^{\infty} t_n x^n \\ &\quad - \frac{t_o}{\log(1-x)}\end{aligned} \tag{6.3.7}$$

Since $\frac{1}{\log(1-x)}$ is of bounded variation in $[\frac{1}{2}, 1)$, it follows from (6.3.5), (6.3.6) and (6.3.7) that

$$s_{n+1} \to \ell \ |L|.$$

The theorem follows now by Theorem 6.3.

THEOREM 6.5

If H_χ *is a regular Hausdorff method and* $s_n \to \ell \ |L|$, *then* $s_n \to \ell \ |LH_\chi|$.

PROOF.

Let h_n and $s(t)$ be as defined in (6.1.3) and (6.2.10) respectively.

Then, by (6.2.11), we have

$$\frac{1}{\log(1+y)} \sum_{n=1}^{\infty} \frac{h_n}{n} \left(\frac{y}{1+y}\right)^n = \int_0^1 \frac{1}{\log(1+y)} s(yt)d\chi(t)$$

$$- s_o \int_0^1 \frac{\log(1+yt)}{\log(1+y)} d\chi(t) + s_o \int_0^1 d\chi(t).$$

Since $s_n \to \ell \ |L|$, we have by Theorem 6.3, that

$$s_{n+1} \to \ell \ |L|,$$

i.e.

$$L(t) = \frac{1}{\log(1+t)} s(t) \quad \text{is of bounded variation in } (0, \infty)$$

Now, we have

$$\frac{1}{\log(1+y)} \sum_{n=1}^{\infty} \frac{h_n}{n} \left(\frac{y}{1+y}\right)^n$$

$$= \int_0^1 \frac{\log(1+yt)}{\log(1+y)} L(yt)d\chi(t) - s_o \int_0^1 \frac{\log(1+yt)}{\log(1+y)} d\chi(t) + s_o \int_0^1 d\chi(t).$$

By Lemma 6.4, $\frac{\log(1+yt)}{\log(1+y)}$ is of uniformly bounded variation and evidently $L(yt)$ is also of uniformly bounded variation in $(0, \infty)$. It follows, by an argument similar to one used in the proof of Theorem 5.3, that

$$h_{n+1} \to \ell \ |L|.$$

Consequently, by Theorem 6.3, we have that

$$h_n \to \ell\ |L|,$$

and the proof of the theorem is complete.

CHAPTER 7

SOME APPLICATIONS

In this chapter some applications of the theorems established in Chapters 5 and 6 are considered and the absolute summability analogues of results obtained by Borwein [7] are proved.

7.1 *DEFINITIONS*

Following Borwein [7], we redefine the method A_λ for $\lambda = -1$ to be the logarithmic method L; i.e.,

$$s_n \to \ell \; (A_{-1})$$

if

$$-\frac{1}{\log(1-x)} \sum_{n=o}^{\infty} \frac{s_n}{n+1} x^{n+1} \to \ell, \text{ as } x \to 1-.$$

The logarithmic-type method (L, α) introduced by Borwein [7], is defined as follows:

Let $\alpha > 0$ and $u_n^\alpha = (C, \alpha)(u_n)$; i.e.,

$$u_n^\alpha = \frac{1}{\varepsilon_n^\alpha} \sum_{r=o}^{n} \varepsilon_{n-r}^{\alpha-1} u_r.$$

The logarithmic-type summability method (L, α).

If

$$-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} u_n^{\alpha} x^{n+1} \to \alpha\ell ,$$

as $x \to 1$ in $(0, 1)$, we say that the sequence $\{s_n\}$ is (L, α)-*convergent* to ℓ and write

$$s_n \to \ell \ (L, \alpha).$$

For $\alpha = 1$, we get the logarithmic method L, so that

$$(L, 1) = L = A_{-1} .$$

For the sequence $\{\mu_n\}$ of real numbers, we denote the Hausdorff matrix generated by it, as well as the Hausdorff method by H. (See Chapter 5. page 65). If $\mu_n \neq 0$, the Hausdorff matrix generated by $\left\{\frac{1}{\mu_n}\right\}$ will be denoted by H^{-1}.
Throughout this chapter, H and K will stand for Hausdorff matrices (and methods). It is well known (See Hardy [14] Chapter XI) that if H and K are Hausdorff matrices generated by the sequences $\{\mu_n\}$ and $\{\nu_n\}$, then HK is a Hausdorff matrix generated by the sequence $\{\mu_n \nu_n\}$, so that $HK = KH$. Further, it is a standard result that, when $\mu_n \neq 0$, $H \Rightarrow K$ if and only if KH^{-1} is regular.

The summability method (C^*, α).

This is defined to be the Hausdorff summability method generated by the sequence $\left\{\frac{1}{\varepsilon_n^{\alpha}}\right\}$ for $\alpha > -1$, and by $\{\varepsilon_n^{-\alpha}\}$

for $\alpha \leq -1$.

It is an immediate consequence that

$$(C^*, \alpha) = \begin{cases} (C, \alpha) & \alpha > -1 \\ (C, -\alpha)^{-1} & \alpha \leq -1. \end{cases}$$

Absolute Regularity.

A summability method P, is called *absolutely regular* if $|C, 0| \Rightarrow |P|$.

7.2

We can now combine Theorems 5.3 and 6.5 , and put here for ready reference:

THEOREM 7.1

If $\lambda \geq -1$ *and* H *is a regular Hausdorff method, then*

$$|A_\lambda| \Rightarrow |A_\lambda H|.$$

THEOREM 7.2

If $\mu_n \neq 0$ *and* $|KH^{-1}| \Rightarrow |A_\lambda|$, *then* $|K| \Rightarrow |A_\lambda H|$.

PROOF.

Suppose that

$$s_n \to \ell \ |K|,$$

and let $t_n = H(s_n)$.

Then, we have that

$$s_n = H^{-1}(t_n) \to \ell \ |K|,$$

and so

$$t_n \to \ell \ |KH^{-1}|.$$

It follows, by hypothesis, that

$$t_n \to \ell \ |A_\lambda|.$$

Consequently

$$s_n \to \ell \ |A_\lambda H|.$$

and the theorem is proved.

THEOREM 7.3

If either (i) H *is regular or (ii)* $\mu_n \neq 0$ *and* $|H^{-1}| \Rightarrow |A_\lambda|$, *then* $A_\lambda H$ *is absolutely regular.*

PROOF.

We first note that A_λ is absolutely regular for $\lambda \geq -1$; For $\lambda > -1$, it is a consequence of the following result of Mishra [23] (Theorem 3):

$$|C, \alpha| \Rightarrow |A_\lambda| \ , \quad (\alpha > -1);$$

and for $\lambda = -1$, we have, by Theorem 6.2, that

$$|C, 0| \Rightarrow |A| \Rightarrow |L|.$$

It follows from Theorem 7.1 that $A_\lambda H$ is absolutely regular when (i) is satisfied.

When (ii) is satisfied, the absolute regularity of $A_\lambda H$ follows from Theorem 7.2.

THEOREM 7.4

If H *is regular, then* $|A_\lambda K| \Rightarrow |A_\lambda KH|$.

PROOF.

Suppose that

$$s_n \to \ell |A_\lambda K|,$$

so that

$$t_n = K(s_n) \to \ell\ |A_\lambda|.$$

Consequently, by Theorem 7.1,

$$t_n \to \ell\ |A_\lambda H|,$$

i.e.

$$s_n \to \ell\ |A_\lambda HK|.$$

The theorem follows as HK = KH.

THEOREM 7.5

If $\mu_n \neq 0$ *and* $H \Rightarrow K$, *then* $|A_\lambda H| \Rightarrow |A_\lambda K|$.

PROOF.

Since KH^{-1} is regular, we have by Theorem 7.4,

$$|A_\lambda H| \Rightarrow |A_\lambda H\ KH^{-1}| = |A_\lambda H\ H^{-1}K| = |A_\lambda K|.$$

This proves the theorem.

An immediate corollary is the following theorem of Mishra [21] (Theorem 1).

COROLLARY 7.5.1

If $\mu > -1$ *and* $\delta > 0$, *then* $|A,\mu| \Rightarrow |A, \mu+\delta|$.

PROOF.

Take $\lambda = 0$, $H = (C, \mu)$ and $K = (C, \mu+\delta)$ in Theorem 7.5.

The next theorem gives a generalization of this corollary.

THEOREM 7.6

If $\alpha > \beta$ *and* γ *is real, then*

$$|C^*, \gamma| \Rightarrow |A_\lambda(C^*, \beta)| \Rightarrow |A_\lambda(C^*, \alpha)|.$$

PROOF.

We have, as noted in the proof of Theorem 7.3, for $\gamma > -1$, $\lambda \geq -1$

$$|C, \gamma| \Rightarrow |A_\lambda|.$$

Also, by Borwein [6] (|IV|)

$$|C^*, \gamma| \Rightarrow |C^*, \gamma+\delta| , \quad (\text{all real } \gamma;\ \delta > 0),$$

and

$$|C^*, \gamma-\beta| \simeq |(C^*, \gamma)(C^*, \beta)^{-1}| .$$

Consequently,

$$|(C^*, \gamma)(C^*, \beta)^{-1}| \Rightarrow |A_\lambda|.$$

Thus, in virtue of Theorem 7.2, we get

$$|C^*, \gamma| \Rightarrow |A_\lambda(C^*, \beta)| .$$

To complete the proof of the theorem, we note that, for $\alpha > \beta$

$$(C^*, \beta) \Rightarrow (C^*, \alpha),$$

and so, by Theorem 7.5,

$$|A_\lambda(C^*,\ \beta)| \Rightarrow |A_\lambda(C^*,\ \alpha)|.$$

The theorem follows.

THEOREM 7.7

If $\beta > -1$ *and* α *is any real number, then*

$$|A_\lambda(C^*,\ \alpha)| \simeq |A_\beta(C^*,\ \alpha+\beta)|.$$

PROOF.

For $\beta > -1$ and $0 < x < 1$, we have the following identity

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = (1-x)^{\beta+1}\sum_{n=0}^{\infty} \varepsilon_n^\beta s_n^\beta x^n .$$

Hence

$$s_n \to \ell\ |A_0|$$

is equivalent to

$$s_n^\beta \to \ell\ |A_\beta|.$$

Consequently if $(C^*,\ \alpha)(s_n) = s_n^{*\alpha}$, then

$$s_n^{*\alpha} \to \ell\ |A_0|$$

is equivalent to

$$(C^*,\ \beta)(C^*,\ \alpha)(s_n) \to \ell\ |A_\beta|,$$

i.e.,

$$|A_0(C^*, \alpha)| = |A_\beta(C^*, \beta)(C^*, \alpha)|.$$

But

$$(C^*, \beta)(C^*, \alpha) \simeq (C^*, \alpha+\beta),$$

and so, in view of Theorem 7.5, we have

$$|A_\beta(C^*, \beta)(C^*, \alpha)| \simeq |A_\beta(C^*, \alpha+\beta)|.$$

It follows that

$$|A_0(C^*, \alpha)| \simeq |A_\beta(C^*, \alpha+\beta)|,$$

and the proof is complete.

If we take $\beta = -\alpha$, we get the following important result:

$$|A_0(C^*, \alpha)| \simeq |A_{-\alpha}| \quad \text{for} \quad \alpha < 1.$$

Consequently:

THEOREM 7.8

If $-1 < \alpha < 1$, *then*

$$|A, \alpha| \simeq |A_{-\alpha}|.$$

THEOREM 7.9

For $\alpha > 0$, $|L, \alpha| \simeq |A_{-1}(C, \alpha-1)|$.

PROOF.

It is immediate from the definition of the method (L, α), that

$$s_n \to \ell \; |L, \alpha|$$

is equivalent to,

$$(n+1)a_n^{\alpha} \to \alpha\ell \; |A_{-1}|.$$

But, for $\alpha > 0$,

$$(n+1)a_n^{\alpha} = \frac{n+1}{n+\alpha} \; \alpha s_n^{\alpha-1} \quad .$$

By Lemma 6.2 , it follows that

$$\frac{n+1}{n+\alpha} \; \alpha s_n^{\alpha-1} \to \alpha\ell \; |A_{-1}|,$$

if and only if

$$s_n^{\alpha-1} \to \ell \; |A_{-1}|.$$

Consequently, $s_n \to \ell \; |L, \alpha|$

if and only if

$$s_n^{\alpha-1} \to \ell \; |A_{-1}|.$$

The theorem follows.

THEOREM 7.10

For $\alpha > \beta > 0$,

$$|A_o(C, \beta)| \Rightarrow |L, \beta| \Rightarrow |L, \alpha| \quad .$$

PROOF.

We have, by Theorem 7.5, Theorem 6.4 , and Theorem 7.9, that,

$$|A_o(C, \beta)| \simeq |A_o(C, 1)(C, \beta-1)| \Rightarrow |A_{-1}(C, \beta-1)| \simeq |L, \beta| \; .$$

Further, by Theorem 7.6 and Theorem 7.9,

$$|L,\ \beta| \approx |A_{-1}(C,\ \beta-1)| \Rightarrow |A_{-1}(C,\ \alpha-1)| \approx |L,\ \alpha|\ .$$

The theorem is proved.

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